Topology and its Applications 156 (2009) 2560–2564

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Topological group criterion for $C(\mathbf{X})$ in compact-open-like topologies, II

R. Ball [a], V. Gochev [b,*], A. Hager [c], S. Zoble [c]

[a] Department of Mathematics, University of Denver, Denver, CO 80208, USA
[b] Department of Mathematics, Trinity College, Hartford, CT 06106, USA
[c] Department of Mathematics and Computer Science, Wesleyan University, Middletown, CT 06459, USA

ARTICLE INFO

Article history:
Received 10 November 2008
Received in revised form 17 January 2009
Accepted 5 February 2009

Dedicated to Neil Hindman, and to his work

MSC:
primary 54C35, 54D20, 22A22, 03E50
secondary 54A10, 06F20, 18A20, 46A40

Keywords:
$C(\mathbf{X})$
Topological group
Čech–Stone compactification
Polish space
Epi-topology
Compact-zero topology
Space with filter
Continuum hypothesis

ABSTRACT

We continue from “part I” our address of the following situation. For a Tychonoff space $\mathbf{Y}$, the “second epi-topology” σ is a certain topology on $C(\mathbf{Y})$, which has arisen from the theory of categorical epimorphisms in a category of lattice-ordered groups. The topology σ is always Hausdorff, and σ interacts with the point-wise addition $+$ on $C(\mathbf{Y})$ as: inversion is a homeomorphism and $+$ is separately continuous. When is $+$ jointly continuous, i.e. σ is a group topology? This is so if $\mathbf{Y}$ is Lindelöf and Čech-complete, and the converse generally fails. We show in the present paper: under the Continuum Hypothesis, for $\mathbf{Y}$ separable metrizable, if σ is a group topology, then $\mathbf{Y}$ is (Lindelöf and) Čech-complete, i.e. Polish. The proof consists in showing that if $\mathbf{Y}$ is not Čech-complete, then there is a family of compact sets in $\beta\mathbf{Y}$ which is maximal in a certain sense.

1. Introduction

This paper is a sequel to [1], to which we refer for elaboration of the details of context and motivation. However, we shall set some notation and sketch the situation leading up to Definition 1.1.

For a Tychonoff space $\mathbf{Y}$: The set $C(\mathbf{Y})$ of continuous real-valued functions on $\mathbf{Y}$ is an abelian group under $(f+g)(y) = f(y) + g(y)$; the group identity is the function constantly zero. The Čech–Stone compactification is $\beta\mathbf{Y}$, and $\mathcal{C}$ denotes the collection of all cozero-sets of $\beta\mathbf{Y}$ which contain $\mathbf{Y}$. We shall reserve the symbol $\mathcal{C}$ for exactly this situation. For $f : \mathbf{Y} \to \mathbf{Z}$, with $\mathbf{Z}$ compact Hausdorff, $\beta f : \beta\mathbf{Y} \to \mathbf{Z}$ is the unique continuous extension. The map $C^*(\mathbf{Y}) \ni f \to \beta f \in C(\beta\mathbf{Y})$ is a group isomorphism. We let $\mathcal{K}(\mathbf{Y}) =: \{K \mid K$ is a compact subset of $\mathbf{Y}\}$. For any family $\mathcal{A}$ of sets $\mathcal{A}_\delta =: \{\bigcap \mathcal{A}' \mid \mathcal{A}'$ is countable subfamily of $\mathcal{A}\}$.

The following discussion synopsizes considerable information from [1] (and see also [2] and [7]).

A space with Lindelöf filter is a pair $(\mathbf{X}, \mathcal{F})$, where $\mathbf{X}$ is a compact Hausdorff space and $\mathcal{F}$ is a filter base of dense cozero-sets in $\mathbf{X}$. We write $(\mathbf{X}, \mathcal{F}) \in |\mathbf{LSpFi}|$. (Our favorite examples are the $(\beta\mathbf{Y}, \mathcal{C})$ above.) Given such $(\mathbf{X}, \mathcal{F})$: Take $S \in \mathcal{F}_\delta$. The family of all

* Corresponding author.
E-mail addresses: rball@du.edu (R. Ball), Vasil.Gochev@trincoll.edu (V. Gochev), ahager@wesleyan.edu (A. Hager), azoble@wesleyan.edu (S. Zoble).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.04.008

$$U(K) =: \{ f \in C(\mathbf{X}) \mid f = 0 \text{ on } K \} \quad (K \in \mathcal{K}(\mathbf{S}))$$

is a basis of neighborhoods of 0 for a Hausdorff group topology σ_S on $C(\mathbf{X})$. We set

$$\sigma^{\mathcal{F}} =: \bigwedge\{\sigma_S \mid S \in \mathcal{F}_\delta\} \quad (\text{meet in the lattice of topologies on } C(\mathbf{X})).$$

This $\sigma^{\mathcal{F}}$ is T_1, inversion $(f \to -f)$ is a homeomorphism, and $+$ is separately continuous. The general question is: When is $+$ jointly continuous? According to 2.5 of [1], this is so if and only if $(\mathbf{X}, \mathcal{F})$ has "the *TGP*" in Definition 1.1 below.

Before getting to that, though, consider a Tychonoff space $\mathbf{Y}$, and $(\beta\mathbf{Y}, \mathcal{C}) \in |\mathbf{LSpFi}|$. We have the topology $\sigma^{\mathcal{C}}$, as above. We also can topologize $C(\mathbf{Y})$ in a similar fashion: For $f \in C(\mathbf{Y}) \subset C(\mathbf{Y}, [-\infty, +\infty])$, consider the extension $\beta f \in C(\beta\mathbf{Y}, [-\infty, +\infty])$. For $S \in \mathcal{C}_\delta$, the family of all

$$U'(K) =: \{ f \in C(\mathbf{Y}) \mid \beta f = 0 \text{ on } K \} \quad (K \in \mathcal{K}(\mathbf{S}))$$

is a basis of neighborhoods of 0 for a Hausdorff group topology, say t_S, on $C(\mathbf{Y})$, and then the topology of the Abstract is

$$\sigma =: \bigwedge\{t_S \mid S \in \mathcal{C}_\delta\} \quad (\text{meet in the lattice of topologies on } C(\mathbf{Y})).$$

Then, *via* the isomorphism $C^*(\mathbf{Y}) \cong C(\beta\mathbf{Y})$, the relative topology $\sigma/C^*(\mathbf{Y})$ becomes exactly the $\sigma^{\mathcal{C}}$ on $C(\beta\mathbf{Y})$. According to 5.5 of [1], σ is a group topology on $C(\mathbf{Y})$ if and only if $\sigma^{\mathcal{C}}$ is a group topology on $C(\beta\mathbf{Y})$.

Thus the question "When is $(C(\mathbf{Y}), +, \sigma)$ a topological group?" has become a particular case of the question "For $(\mathbf{X}, \mathcal{F}) \in |\mathbf{LSpFi}|$, when is $(C(\mathbf{X}), +, \sigma^{\mathcal{F}})$ a topological group?", which, as we said, happens if and only if the *TGP* in the following obtains.

Definition 1.1. Let $(\mathbf{X}, \mathcal{F}) \in |\mathbf{LSpFi}|$.

The family $\mathfrak{L}$ of subsets of $\mathbf{X}$ is called adequate if $[\mathfrak{L} \subseteq \mathcal{K}(\mathbf{X})$ and $\mathfrak{L} \cap \mathcal{K}(\mathbf{S}) \neq \emptyset\ \forall \mathbf{S} \in \mathcal{F}_\delta]$.

For adequate $\mathfrak{L}, \mathfrak{M}$, $\mathfrak{L} \overset{z}{\prec} \mathfrak{M}$ means: For each $M_1, M_2 \in \mathfrak{M}$ and zero-sets $Z_i \supseteq M_i$, there is an $L \in \mathfrak{L}$ with $L \subseteq Z_1 \cap Z_2$. ("Adequate" refers to the filter $\mathcal{F}$. If necessary, we shall say "$\mathcal{F}$-adequate".)

$(\mathbf{X}, \mathcal{F})$ has the Topological Group Property *TGP* if $[\forall$ adequate $\mathfrak{L}\ \exists$ adequate $\mathfrak{M}$ with $\mathfrak{L} \overset{z}{\prec} \mathfrak{M}]$. Thus, $(\mathbf{X}, \mathcal{F})$ fails the *TGP* if and only if there is adequate $\mathfrak{L}$ which is maximal with respect to $\overset{z}{\prec}$.

(The Hausdorff property deserves comment. For a general $(\mathbf{X}, \mathcal{F})$, the topology $\sigma^{\mathcal{F}}$ on $C(\mathbf{X})$ need not be Hausdorff: an example in 6.5 of [2] can be adapted easily. However, by 2.3 of [1], if $\bigcap \mathcal{F}$ is dense in $\mathbf{X}$, then $\sigma^{\mathcal{F}}$ is Hausdorff. For the "favorite examples" $(\beta\mathbf{Y}, \mathcal{C})$, we have $\bigcap \mathcal{C} = \upsilon\mathbf{Y}$, the Hewitt realcompactification of $\mathbf{Y}$ [8], so $\sigma^{\mathcal{C}}$ on $C(\beta\mathbf{Y})$ is Hausdorff, and it follows easily that the topology σ on $C(\mathbf{Y})$ of the preceding paragraph is also Hausdorff. See [2], Section 6 for further discussion.)

We now summarize the results of our earlier attack [1] on the question [What are the $\mathbf{Y}$ for which $(\beta\mathbf{Y}, \mathcal{C})$ has the *TGP*?].

$\mathbf{Y}$ is called Čech-complete if $\mathbf{Y}$ is G_δ in $\beta\mathbf{Y}$ [5]. It follows that $\mathbf{Y}$ is Lindelöf and Čech-complete if and only if $\mathbf{Y} \in \mathcal{C}_\delta$. (The implication $\Rightarrow$ uses [5], 3.12.25, and $\Leftarrow$ uses [5], 3.8.F(b).) Let $\mathbf{D}$ be the discrete space of power ω_1. Let $\lambda\mathbf{D}$ be $\mathbf{D}$ with one point adjoined, whose neighborhoods have countable complement; $\lambda\mathbf{D}$ is a P-space, which is Lindelöf, not Čech-complete. Note that $(\beta\mathbf{Y}, \mathcal{C}) = (\beta\upsilon\mathbf{Y}, \mathcal{C})$ and $C(\mathbf{Y}) \cong C(\upsilon\mathbf{Y})$. (See [8].)

Theorems 1.2. *([1], 1.2, 1.3 and 1.4)*

(1) *If $\upsilon\mathbf{Y}$ is Lindelöf and Čech-complete, then $(\beta\mathbf{Y}, \mathcal{C})$ has the TGP.*
(2) *$(\beta\lambda\mathbf{D}, \mathcal{C})$ has the TGP.*
(3) *$(\beta\mathbf{D}, \mathcal{C})$ fails the TGP. Suppose $\mathbf{Y}$ is paracompact, locally compact, zero-dimensional. If $(\beta\mathbf{Y}, \mathcal{C})$ has the TGP then $\mathbf{Y}$ is Lindelöf.*

Here (1) is elementary from the definition of $\sigma^{\mathcal{F}}$, (2) and (3) require considerable work, and seem to be ultimately set-theoretic.

(2) above says that the converse to (1) fails. In the theorem of this paper (Section 2), we show that this converse holds within the class of separable metrizable spaces, assuming the Continuum Hypothesis [CH].

Questions about Theorem 1.2 remain: Is (2) true replacing $\lambda\mathbf{D}$ by any Lindelöf P-space? Is it true that $[(\beta\mathbf{Y}, \mathcal{C})$ has the *TGP* $\Rightarrow \upsilon\mathbf{Y}$ Lindelöf]?

2. The theorem

The theorem in the Abstract is equivalent, *via* the discussion in Section 1, to the following.

Theorem 2.1 *([CH]). Suppose $\mathbf{Y}$ is separable metrizable (thus Lindelöf). If $(\beta\mathbf{Y}, \mathcal{C})$ has the TGP, then $\mathbf{Y}$ is Čech-complete.*

A difficulty in proving Theorem 2.1 is coping with the zero-sets of $\beta\mathbf{Y}$, $Z_i \supseteq M_i$ in the definition of $\mathfrak{L} \overset{z}{\prec} \mathfrak{M}$ in Definition 1.1. This will be circumvented by (i) passing to a metrizable compactification $\mathbf{X}$ of separable metrizable $\mathbf{Y}$ (*via* Urysohn's Metrization procedure [5]), (ii) noting that in such $\mathbf{X}$, every closed set is a zero-set, so that $\mathfrak{L} \overset{z}{\prec} \mathfrak{M}$ takes the simpler form $[\forall M_1, M_2 \in \mathfrak{M}\ \exists L \in \mathfrak{L}\ (L \subseteq M_1 \cap M_2)]$, and (iii) proving the following.

Lemma 2.2. *Suppose* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are compactifications of* $\mathbf{Y}$, $\mathcal{C}_i$ *is the family of cozero-sets of* $\mathbf{X}_i$ *containing* $\mathbf{Y}$; *so* $(\mathbf{X}_i, \mathcal{C}_i) \in |\mathbf{LSpFi}|$. *Suppose there is continuous* $\mathbf{X}_1 \overset{\lambda}{\twoheadrightarrow} \mathbf{X}_2$ *extending the identity map on* $\mathbf{Y}$ *(i.e.,* $\mathbf{X}_1 \geqslant \mathbf{X}_2$ *as compactifications).*

Suppose $\mathbf{Y}$ *is Lindelöf. If* $(\mathbf{X}_2, \mathcal{C}_2)$ *fails the TGP, then so does* $(\mathbf{X}_1, \mathcal{C}_1)$.

Proof. Suppose $\mathbf{X}_1 \overset{\lambda}{\twoheadrightarrow} \mathbf{X}_2$ and $\mathcal{C}_i$ are as above (not yet assuming $\mathbf{Y}$ is Lindelöf). Note that $\lambda\lambda^{-1}(B) = B$ for any $B \subseteq \mathbf{X}_2$, since λ is a surjection, and $\lambda(\mathbf{X}_1 - \mathbf{Y}) = \mathbf{X}_2 - \mathbf{Y}$, since λ extends the identity [8]. For $\mathfrak{A}_i$ a family of subsets of $\mathbf{X}_i$, let $\lambda(\mathfrak{A}_1) \equiv \{\lambda(A) | A \in \mathfrak{A}_1\}$ and $\lambda^{-1}(\mathfrak{A}_2) \equiv \{\lambda^{-1}(A) | A \in \mathfrak{A}_2\}$. Let "$\mathfrak{L}_i$ is $\mathcal{C}_i$-adequate", "$\mathfrak{L}_i \overset{z}{\prec} \mathfrak{M}_i(\mathcal{C}_i)$", and "$\mathfrak{L}_i$ is $\overset{z}{\prec}$-maximal $(\mathcal{C}_i)$" have the obvious meanings.

(1) If $\mathfrak{L}_1 (\subseteq \mathcal{K}(\mathbf{X}_1))$ is $\mathcal{C}_1$-adequate, then $\lambda(\mathfrak{L}_1)$ is $\mathcal{C}_2$-adequate.
(2) If $\mathfrak{L}_1 \overset{z}{\prec} \mathfrak{M}_1(\mathcal{C}_1)$, then $\lambda(\mathfrak{L}_1) \overset{z}{\prec} \lambda(\mathfrak{M}_1)(\mathcal{C}_2)$.

The proofs of (1) and (2) are routine calculations. Note that (1) is needed for (2): $\overset{z}{\prec}$ is only defined for adequate families.

Now assume $\mathbf{Y}$ is Lindelöf, then the following statements hold.

(3) $\forall S \in (\mathcal{C}_1)_\delta\ \exists T \in (\mathcal{C}_2)_\delta$ with $\lambda^{-1}(T) \subseteq S$.
(4) If $\mathfrak{L}_2 (\subseteq \mathcal{K}(\mathbf{X}_2))$ is $\mathcal{C}_2$-adequate, then $\lambda^{-1}(\mathfrak{L}_2)$ is $\mathcal{C}_1$-adequate.
(5) If $\mathfrak{L}_2$ is $\overset{z}{\prec}$-maximal $(\mathcal{C}_2)$, then $\lambda^{-1}(\mathfrak{L}_2)$ is $\overset{z}{\prec}$-maximal $(\mathcal{C}_1)$.

The lemma follows from (5). We prove (3), (4), and (5).

Proof of (3). Let $S \in (\mathcal{C}_1)_\delta$, so $S = \bigcap S_n$ for S_n's cozero. $\lambda(\mathbf{X}_1 - S_n)$ is closed, disjoint from $\mathbf{Y}$. By Smirnov's Theorem on "normal placement" ([5], 3.12.25), there is $T_n \in \mathcal{C}_2$ with $\mathbf{Y} \subseteq T_n \subseteq \mathbf{X}_2 - \lambda(\mathbf{X}_1 - S_n)$. It follows that $\mathbf{Y} \subseteq \lambda^{-1}(T_n)$, so $T \equiv \bigcap T_n \in (\mathcal{C}_2)_\delta$ and $\lambda^{-1}(T) \subseteq S$.

Proof of (4). For $S \in (\mathcal{C}_1)_\delta$, take T *per* (3). If $\mathfrak{L}_2$ is $\mathcal{C}_2$-adequate, then there exists $L \in \mathfrak{L}_2 \cap \mathcal{K}(T)$, so $\lambda^{-1}(L) \in \lambda^{-1}(\mathfrak{L}_2) \cap \mathcal{K}(\lambda^{-1}(T)) \subseteq \lambda^{-1}(\mathfrak{L}_2) \cap \mathcal{K}(S)$.

Proof of (5). Suppose $\mathfrak{L}_2$ is $\overset{z}{\prec}$-maximal $(\mathcal{C}_2)$. By (4), $\lambda^{-1}(\mathfrak{L}_2)$ is $\mathcal{C}_1$-adequate, and we can address the question $[\exists \mathfrak{M}_1\ (\lambda^{-1}(\mathfrak{L}_2) \overset{z}{\prec} \mathfrak{M}_1)?]$. If there were such $\mathfrak{M}_1$, then $\mathfrak{L}_2 = \lambda\lambda^{-1}(\mathfrak{L}_2) \overset{z}{\prec} \lambda(\mathfrak{M}_1)$ by (2); so there is no such $\mathfrak{M}_1$. $\square$

For $\mathbf{Y}$ separable metrizable: for any metrizable compactification $\mathbf{X}$ of $\mathbf{Y}$, there is the $\beta\mathbf{Y} \overset{\lambda}{\twoheadrightarrow} \mathbf{X}$ as in Lemma 2.2, and $\mathbf{Y}$ is Čech-complete if and only if $\mathbf{Y}$ is G_δ in $\mathbf{X}$ ([5], 3.9.1). So Lemma 2.2 and the following more general result prove Theorem 1.2.

Theorem 2.3 *([CH]). Let* $\mathbf{X}$ *be compact metrizable with dense subset* $\mathbf{Y}$. *Let* $\mathcal{J}$ *stands for the family of all cozero (= open) sets in* $\mathbf{X}$ *which contain* $\mathbf{Y}$.

If $\mathbf{Y}$ *is not* G_δ *in* $\mathbf{X}$, *then* $(\mathbf{X}, \mathcal{J})$ *fails the TGP: there is adequate* $\mathfrak{L}_0$ *which is* $\prec$*-maximal (=* $\overset{z}{\prec}$*-maximal). That is, if* $(\mathbf{X}, \mathcal{J})$ *has the TGP, then* $\mathbf{Y}$ *is Čech-complete.*

We require two simple lemmas. A regular closed set in a space is a subset which is the closure of its interior. $CND(\mathbf{X})$ is the family of closed nowhere dense subsets of $\mathbf{X}$.

Lemma 2.4. *Suppose* $\mathbf{X}$ *is a compactification of* $\mathbf{Y}$. *Then,* $\overline{\mathbf{X} - \mathbf{Y}}$ *is a regular closed set in* $\mathbf{X}$.

Proof. First, if U is open in $\mathbf{Y}$ with $\overline{U}^{\mathbf{Y}}$ compact, then $\overline{U}^{\mathbf{Y}} = \overline{U}^{\mathbf{X}}$, and since $\mathbf{Y}$ is dense in $\mathbf{X}$, U is open in $\mathbf{X}$. Now consider the set of locally compact points

$$lc\,\mathbf{Y} = \{p \in \mathbf{Y} \mid \exists U \text{ open in } \mathbf{Y} \text{ with } p \in U,\ \overline{U}^{\mathbf{Y}} \text{ compact}\}.$$

Then, $lc\,\mathbf{Y}$ is open in $\mathbf{X}$, $lc\,\mathbf{Y} \cap \overline{\mathbf{X} - \mathbf{Y}} = \emptyset$, and $\mathbf{X} = lc\,\mathbf{Y} \cup \overline{\mathbf{X} - \mathbf{Y}}$. This implies the result, since whenever (any) $\mathbf{X} = G \cup F$, G open and F closed and $G \cap F = \emptyset$, then $int\,F = \mathbf{X} - \overline{G}$. $\square$

Lemma 2.5. *Suppose* $\mathbf{X}$ *is any space, and* T *is closed in* $\mathbf{X}$. *The following are equivalent.*

(i) *T is regular closed in* $\mathbf{X}$.
(ii) *If S is dense in* $\mathbf{X}$ *(or, dense open, or dense G_δ), then $S \cap T$ is dense in T.*
(iii) *If $E \in CND(\mathbf{X})$, then $E \cap T \in CND(T)$.*

This proof is easy and omitted.

Proof of Theorem 2.3. Let $G_\delta(\mathbf{X}, \mathbf{Y})$ be the set of all G_δ's in $\mathbf{X}$ which contain $\mathbf{Y}$, suppose $\mathbf{X}$ is compact metrizable, and $\mathbf{Y}$ is dense and not G_δ in $\mathbf{X}$. Then $|\mathbf{X} - \mathbf{Y}| \geqslant \omega$ and $|G_\delta(\mathbf{X}, \mathbf{Y})|$ and $|CND(\mathbf{X})|$ are each $\geqslant 2^\omega$. But the G_δ's and CND's are Borel sets, and there are only 2^ω Borel sets. (See [4], 8.5. This reference is to Baire sets. In a metrizable space, Baire = Borel.) Thus $|G_\delta(\mathbf{X}, \mathbf{Y})| = 2^\omega = |CND(\mathbf{X})|$.

Now take enumerations

$$G_\delta(\mathbf{X}, \mathbf{Y}) = \{S_\alpha \mid \alpha < 2^\omega\} \quad \text{and} \quad CND(\mathbf{X}) = \{E_\alpha \mid \alpha < 2^\omega\},$$

and let $T = \overline{\mathbf{X} - \mathbf{Y}}$. By Lemma 2.4, T is regular closed.

Suppose $\mathbf{Y}$ is not G_δ in $\mathbf{X}$. Then $\mathbf{Y} \cap T$ is not G_δ in $\mathbf{X}$ (since $\mathbf{Y} = (\mathbf{Y} \cap T) \cup (\mathbf{X} - T)$), thus not G_δ in T (since a G_δ in a G_δ is a G_δ).

Let $\alpha < 2^\omega$. There are

$$p_\alpha \in S_\alpha \cap T - \bigcup_{\beta<\alpha} (E_\beta \cap T) \quad \text{and} \quad q_\alpha \in S_\alpha \cap T - \mathbf{Y} \cap T.$$

(There is p_α since: Each $E_\beta \cap T \in CND(T)$, by Lemma 2.5, so under [CH], their union is meagre in T. But $S_\alpha \cap T$ is G_δ in T, thus not meagre in T by the Baire Category Theorem. There is q_α since $S_\alpha \cap T \supseteq \mathbf{Y} \cap T$ with the former dense G_δ in T, by Lemma 2.5, and the latter not G_δ in T.)

Let $\mathfrak{L}_0 = \{\{p_\alpha, q_\alpha\} \mid \alpha < 2^\omega\}$. This is evidently adequate, and we now show $\mathfrak{L}_0$ is $\prec$-maximal.

Take countable F dense in $\mathbf{X} - \mathbf{Y}$, so $\mathbf{X} - F \in G_\delta(\mathbf{X}, \mathbf{Y})$ and there is $\gamma_1 < 2^\omega$ with $\mathbf{X} - F = S_{\gamma_1}$.

Suppose $\mathfrak{M}$ is adequate.

(i) There is $M_1 \in \mathfrak{M}$ with $M_1 \subseteq S_{\gamma_1}$. Then, $M_1 \cap T \in CND(\mathbf{X})$. (If there is open nonvoid $U \subseteq M_1 \cap T$, then $U \subseteq T$ so $U \cap (\mathbf{X} - \mathbf{Y}) \neq \emptyset$, so $\emptyset \neq U \cap F \subseteq M_1 \subseteq S_{\gamma_1} = \mathbf{X} - F$. Contradiction.) So there is $\gamma_2 < 2^\omega$ with $M_1 \cap T = E_{\gamma_2}$. Consequently, for $\alpha > \gamma_2$, we have $p_\alpha \notin E_{\gamma_2}$, so $p_\alpha \notin M_1$ (since $p_\alpha \in T$).

(ii) Let $S = \bigcap_{\alpha<\gamma_2} (\mathbf{X} - \{q_\alpha\})$. Under [CH], there is γ_3 with $S = S_{\gamma_3}$, and there is $M_2 \in \mathfrak{M}$ with $M_2 \subseteq S_{\gamma_3}$. Thus, for $\alpha \leqslant \gamma_2$, we have $q_\alpha \notin M_2$ (since $M_2 \subseteq S_{\gamma_3}$).

So, for every $\alpha < 2^\omega$, $\{p_\alpha, q_\alpha\} \nsubseteq M_1 \cap M_2$, and $\mathfrak{L}_0 \not\prec \mathfrak{M}$. $\square$

Remarks 2.6. In the proof of Theorem 2.3 above, the step "$\exists p_\alpha$" requires only the axiom $[p = c]$, which is weaker than [CH]; see [6]. But the final step (ii) seems to need [CH].

We do not know if [CH] is actually required for Theorem 2.3 (or for the assertion in Theorem 2.3 using simply $\mathbf{X} = [0, 1]$, $\mathbf{Y} = \mathbb{Q} \cap [0, 1]$, for example). (Note that $\mathbb{Q} \cap [0, 1]$ is not Čech-complete ([5], 3.9.B).)

3. Some remarks

We comment on various aspects of the situation.

3.1. About Lemma 2.2

(a) First, if $\mathbf{X}_1 \overset{\lambda}{\twoheadrightarrow} \mathbf{X}_2$ is any continuous surjection, a group embedding $C(\mathbf{X}_2) \xrightarrow{\widetilde{\lambda}} C(\mathbf{X}_1)$ is defined by $\widetilde{\lambda}(f) = f \circ \lambda$. If λ is exactly as in Lemma 2.2, then $\lambda^{-1}(\mathcal{C}_2) \subseteq \mathcal{C}_1$ and $(C(\mathbf{X}_2), \sigma^{\mathcal{C}_2}) \xrightarrow{\widetilde{\lambda}} (C(\mathbf{X}_1), \sigma^{\mathcal{C}_1})$ is a topological embedding. (The proof is much as the proof of Lemma 2.2 but requiring details about the $\sigma^{\mathcal{C}}$'s). So if $(C(\mathbf{X}_2), +, \sigma^{\mathcal{C}_2})$ is not a topological group, then neither is $(C(\mathbf{X}_1), +, \sigma^{\mathcal{C}_1})$. That is a "better version of Lemma 2.2" which we omit explaining fully since we have omitted all details about the $\sigma^{\mathcal{F}}$'s.

(b) The information in (a) has a natural generalization. Suppose $(\mathbf{X}_i, \mathcal{F}_i) \in |\mathbf{LSpFi}|$ and $\mathbf{X}_1 \overset{\lambda}{\twoheadrightarrow} \mathbf{X}_2$ is continuous and $\lambda^{-1}(\mathcal{F}_2) \subseteq \mathcal{F}_1$. Then λ is a morphism of the category **SpFi** (by definition), and $(C(\mathbf{X}_2), \sigma^{\mathcal{F}_2}) \xrightarrow{\widetilde{\lambda}} (C(\mathbf{X}_1), \sigma^{\mathcal{F}_1})$ is continuous (it can be shown). If further $[\forall S \in (\mathcal{F}_1)_\delta\ \exists T \in (\mathcal{F}_2)_\delta$ with $\lambda^{-1}(T) \subseteq S]$ (exactly (3) in the proof of Lemma 2.2), then $\widetilde{\lambda}$ is a topological embedding.

3.2. About Theorem 2.3

The proof of Theorem 2.3 actually proves the following. Suppose $\mathbf{X}$ is compact and $\mathbf{Y}$ is dense in $\mathbf{X}$, that $|G_\delta(\mathbf{X}, \mathbf{Y})| = 2^\omega = |CND(\mathbf{X})|$, that $\mathbf{X} - \mathbf{Y}$ has a countable dense set, and that $\overline{\mathbf{X} - \mathbf{Y}}$ is G_δ in $\mathbf{X}$. Then, if $\mathbf{Y}$ is not G_δ in $\mathbf{X}$, then there is adequate $\mathfrak{L}_0$ which is $\prec$-maximal. Here, "adequate", etc., are defined *mutatis mutandis*, with respect to the filter base of all dense open sets in $\mathbf{X}$ which contain $\mathbf{Y}$.

The previous paragraph is speaking of **SpFi** – spaces with filters – not **LSpFi** – spaces with Lindelöf filter – and we lose contact with the motivation and the details about the topologies from [2] and [7]. So, while we do not know what the above means, we note that the theory of **SpFi** is, roughly, the theory of completely regular frames [3], and the latter has been much studied (e.g., [9]).

3.3. First epi-topology

Referring to the discussion in Section 1 about $\sigma^{\mathcal{C}}$ versus the "second epi-topology" σ on $C(\mathbf{Y})$, there is a "first epi-topology" from [2] τ on $C(\mathbf{Y})$, and its companion $\tau^{\mathcal{C}}$ on $C(\beta\mathbf{Y})$, and for any $(\mathbf{X},\mathcal{F}) \in |\mathbf{LSpFi}|$, the more general $\tau^{\mathcal{F}}$ on $C(\mathbf{X})$. [1] deals with both $\sigma^{\mathcal{F}}$ and $\tau^{\mathcal{F}}$.

The analogue for $\tau^{\mathcal{F}}$ of the *TGP* is: for every adequate $\mathfrak{L}$ there is adequate $\mathfrak{M}$ with $\mathfrak{L} \overset{o}{\prec} \mathfrak{M}$, where $\overset{o}{\prec}$ means "in $\overset{z}{\prec}$, replace the zero sets by open sets". Then, Theorem 1.2 here is true also of $\tau^{\mathcal{F}}$. However, we do not know if Theorem 2.3 here is true using $\overset{o}{\prec}$.

3.4. Several questions

We collect some of the questions which we have not answered.

(1) $(\beta\mathbf{Y},\mathcal{C})$ has the *TGP* $\overset{?}{\Rightarrow}$ $\upsilon\mathbf{Y}$ Lindelöf? Cf. Theorem 1.2(3).

(2) $\mathbf{Y}$ a Lindelöf P-space $\overset{?}{\Rightarrow}$ $(\beta\mathbf{Y},\mathcal{C})$ has the *TGP*? Cf. Theorem 1.2(1).

(3) Assume the setting of Theorem 2.3. Then Theorem 2.3 can be put: [CH] $\Rightarrow \exists\mathfrak{L}_0$ $\prec$-maximal. Does the converse hold? Or, $\exists\mathfrak{L}_0$ $\prec$-maximal $\overset{?}{\Rightarrow}$ [CH], or Martin's Axiom, or $[p = c]$? Or, the same questions just using $\mathbf{X} = [0,1]$ and $\mathbf{Y} = \mathbb{Q} \cap [0,1]$.

(4) Questions (1), (2), (3) using $\overset{o}{\prec}$ instead of $\overset{z}{\prec}$. See 3.3 above.

(5) Do Theorem 2.3 and Theorem 2.1 hold using $\overset{o}{\prec}$ instead of $\overset{z}{\prec}$?

Questions (4) and (5) reflect on the topologies $\tau^{\mathcal{F}}$, $\tau^{\mathcal{C}}$, τ mentioned in 3.3 above.

Acknowledgement

We are grateful to the referee for a very careful and thoughtful reading of the paper, and for numerous suggestions which have improved the paper.

References

[1] R. Ball, V. Gochev, A. Hager, S. Todorčević, S. Zoble, Topological group criterion for $C(X)$ in compact-open-like topologies, I, Topology Appl. 156 (2009) 710–720.
[2] R. Ball, A. Hager, Epi-topology and epi-convergence for archimedean lattice-ordered groups with weak unit, Appl. Categ. Structures 15 (2007) 81–107.
[3] R. Ball, A. Hager, A. Molitor, Spaces with filters, in: C. Gilmour, B. Banaschewski, H. Herrlich (Eds.), Proc. Symp. Cat. Top., Univ. Cape Town, 1994, Dept. Math. and Appl. Math., Univ. Cape Town, 1999, pp. 21–36.
[4] W. Comfort, S. Negrepontis, Continuous Pseudometrics, Lecture Notes, vol. 14, Dekker, 1975.
[5] R. Engelking, General Topology, Heldermann, 1989.
[6] D. Fremlin, Consequences of Martin's Axiom, Cambridge Univ. Press, 1984.
[7] V. Gochev, Monomorphisms in spaces with Lindelöf filters via some compact-open-like topologies on $C(X)$, submitted for publication.
[8] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand, 1960, reprinted as: Grad. Texts, vol. 43, Springer-Verlag, 1976.
[9] P. Johnstone, Stone Spaces, Cambridge Univ. Press, 1982.