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On the extender algebra being complete

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We show that a Woodin cardinal is necessary for the extender algebra to be complete. Our proof is relatively simple and does not use fine structure.

1 Introduction

Let $B_{\delta,\omega}$ denote the algebra of equivalence classes of Boolean strings in V_δ formed from a countable set of atoms $\{a_n \mid n < \omega\}$. Two strings φ and ψ are equivalent in $B_{\delta,\omega}$ if $A(\varphi) = A(\psi)$ in all generic extensions, where the map $A : B_{\delta,\omega} \longrightarrow P(P(\omega))$ is defined inductively from

$$A(a_n) = \{x \subseteq \omega \mid n \in x\}.$$

Let δ be an inaccessible cardinal and $\mathcal{E} \subset V_\delta$ a set of extenders. The extender algebra $W_\delta(\mathcal{E})$ is defined as a quotient of $B_{\delta,\omega}$ by an ideal $I_{\mathcal{E}}$ defined from $\mathcal{E}$. For simplicity we require that an $E \in \mathcal{E}$ is a (κ, λ)-extender for some κ and λ, that λ is inaccessible and that E is strong to λ. $I_{\mathcal{E}}$ consists of elements of $B_{\delta,\omega}$ represented by strings of the form

$$i_E(\vee_{\alpha<\kappa}\varphi_\alpha) \wedge \neg(\vee_{\alpha<\kappa}\varphi_\alpha)$$

for such an $E \in \mathcal{E}$ and sequence $\{\varphi_\alpha \mid \alpha < \kappa\}$ from V_κ. The key property of $W_\delta(\mathcal{E})$ if δ is a Woodin cardinal as witnessed by $\mathcal{E}$ is that it satisfies the δ-chain condition, equivalently that it is a complete Boolean algebra. From this alone Woodin deduces the "every real generic" theorem, one of the most important tools in inner model theory.

Theorem 1.1 (Every real generic; Woodin) *Suppose M is a countable premouse and $\mathcal{E} \subset V_\delta^M$ is a set of extenders in M.*

1. *If δ is Woodin in M as witnessed by $\mathcal{E}$, then $W_\delta(\mathcal{E})$ is a complete Boolean algebra in M.*
2. *If $W_\delta(\mathcal{E})$ is complete in M, and M is $(\omega_1 + 1)$-iterable, then for every real $x \subset \omega$ in V there is an iteration $i : M \longrightarrow M^*$ via a tree of countable length such that x is M^*-generic for $i(W_\delta(\mathcal{E}))$.*

Here a premouse is merely a transitive set M and an ordinal $\delta \in M$ which satisfies a certain fragment of ZFC (see [2]). By M^*-generic we mean that $G_x = \{[\varphi] \in i(W_\delta(\mathcal{E})) \mid x \in A(\varphi)\}$ is an M^*-generic filter. This will obtain exactly when

$$x \notin \bigcup_{[\varphi]\in i(I_{\mathcal{E}})} A(\varphi)$$

in which case $M^*[x] = M^*[G_x]$. The reader is referred to [3, Theorem 7.14] for more details. The proof given there is easily modified to prove the version above, that is, it is not necessary for M to be a fine structural model, a fact well-known to inner model theorists. The purpose of this note is to prove a converse to Theorem 1.1.

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Theorem 1.2 *Assume δ is an inaccessible cardinal and $V_\delta^\#$ exists. Assume $\mathcal{E} \subset V_\delta$ is a set of extenders and $W_\delta(\mathcal{E})$ is complete. Then some ordinal $\alpha \le \delta$ is a Woodin cardinal in $L(V_\alpha)$.*

As a corollary we have that if δ is least such that some $W_\delta(\mathcal{E})$ is complete in $L(V_\delta)$, then δ is Woodin in $L(V_\delta)$. Woodin had proved an identical result using the backgrounded $L[\mathcal{E}]$ construction of [1]. Our proof has the virtue of only using "coarse methods", that is, techniques and results from [2]. Both are not optimal in that they seem to require the existence of $V_\delta^\#$. Moreover, it seems likely that there is a direct equivalence between the completeness of $W_\delta(\mathcal{E})$ and δ being Woodin in V. Here is one version of this conjecture.

Question 1.3 *Suppose δ is a cardinal and $\mathcal{E} \subset V_\delta$ is a set of extenders each of which is strong to its length. Suppose $W_\delta(\mathcal{E})$ is complete and has size δ. Must δ be a Woodin cardinal?*[1)]

Our argument uses two theorems from [2]. The first, a slight variation on their Corollary 5.11, concerns iterability of countable submodels of the universe under a smallness assumption, and the second shows that complicated iteration trees actually give rise to Woodin cardinals.

Theorem 1.4 (Iterability; Martin, Steel) *Suppose no $\alpha \le \delta$ is Woodin in $L(V_\alpha)$. Suppose $\pi : N \longrightarrow V_\theta$ is elementary with N countable. Let M be the preimage of $L(V_\delta)$. Then M is ω_1-iterable via the strategy of picking the unique cofinal well-founded branch at limit stages.*

For an iteration tree T of limit length λ on a premouse M define

$$\delta(T) = \sup_{\alpha<\lambda} \inf_{\alpha\le\gamma<\lambda} \operatorname{str}(E_\gamma^T)$$

as in [2]. If b and c are cofinal branches of T, then $V_{\delta(T)}^{M_b} = V_{\delta(T)}^{M_c}$ is well-founded and we call this the "common part" of T and denote it $M(T)$. The following is [2, Corollary 2.3].

Theorem 1.5 (Distinct branches; Martin, Steel) *Suppose T is an iteration tree of limit length λ on a premouse M and b and c are distinct cofinal branches. Suppose $\alpha \ge \delta(T)$ belongs to $\operatorname{wfp}(M_b) \cap \operatorname{wfp}(M_c)$. Then $L_\alpha(M(T))$ thinks that $\delta(T)$ is a Woodin cardinal.*

2 Proof of Theorem 1.2

Suppose $V_\delta^\#$ exists and $W_\delta(\mathcal{E})$ is complete. Then $W_\delta(\mathcal{E})$ is complete in $L(V_\delta)$.[2)] We assume that no $\alpha \le \delta$ is Woodin in $L(V_\alpha)$. Let θ be sufficently large and $\pi : N \longrightarrow V_\theta$ be elementary with N countable and transitive and $V_\delta^\#, \mathcal{E}$ in the range of π. Let M denote the collapse of $L(V_\delta)$, that is $M = L^N(\pi^{-1}(V_\delta))$, and let $\bar\delta$ denote the preimage of δ. Then M is ω_1-iterable by Theorem 1.4. The (unique) iteration strategy is to pick the unique cofinal well-founded branch at every limit stage. We will work inside of $M[g]$, where $g \subset \operatorname{Col}(\omega, V_{\bar\delta}^M)$ is V-generic. It is easy to see that $\operatorname{Col}(\omega, V_{\bar\delta}^M)$ is $\bar\delta^+$-cc in M and that $M[g] = L_\eta[g]$, where $\eta = M \cap \mathrm{OR}$. It follows that $M[g]$ satisfies the Axiom of Choice (M may not), a fact which we shall use later. Our use of the existence of $V_\delta^\#$ is the following.

Claim 2.1 *For any $\gamma < \eta$ and $M[g]$-generic $h \subset \operatorname{Col}(\omega, \gamma)$ the model $M[g][h]$ is Σ_2^1 correct in V.*

Proof. M is of the form $L_\eta(V_{\bar\delta}^M)$. By standard facts about sharps, $\pi^{-1}(V_\delta^\#) = \pi^{-1}(V_\delta)^\#$. Thus η, which is the ordinal height of N as well as M, is a limit cardinal of $L(M)$, in fact an inaccessible cardinal of $L(M)$ (we use here that θ is a suitable reflection point). Thus g is generic over $L(M)$, h is generic over $L(M)[g]$, and η is a limit cardinal of $L(M)[g][h]$. It follows from Schonfield absoluteness that $M[g][h]$ is Σ_2^1-correct in V. □

We will work inside $M[g]$, where $g \subset \operatorname{Col}(\omega, V_{\bar\delta})$ is a fixed M-generic in V. Let $x \in M[g]$ be a real coding the generic g and $V_{\bar\delta}^M$ in the sense that $L[x] = L[g]$, say $x = \{(n, m) \mid g(n) \in g(m)\}$. In $M[g]$ we construct an iteration tree on M to make x generic over the final model. Note that $(\omega_1)^{M[g]} = (\bar\delta^+))^M < \eta$. Call this ordinal κ. We claim that either

1. at some limit stage $\alpha \le \kappa$, $M[g]$ does not see a cofinal well-founded branch of $T \restriction \alpha$, or

2. the construction succeeds in producing an iterate M^* over which x is generic via a tree which is countable in $M[g]$.

1) One could also consider the δ-generator version of the extender algebra; see [3].

2) $L(V_\delta)$ may not see an arbitrary $\mathcal{E}$ but we may assume without loss of generality that $\mathcal{E}$ is the set of all extenders in V_δ.

This proof of Theorem 1.1. shows that 2. must hold if 1. fails. There are some subtle issues here so we elaborate on this point. First, we have shown that $M[g]$ is a model of ZFC. In fact, $M[g] = L_\eta[g]$ has a definable well-ordering. This ordering is used to construct the tree T on M. At a stage β find the least extender E which belongs to $i_{0,\beta}(\mathcal{E})$ (which is a subset of M_β and hence $M[g]$) and which generates an element $[\varphi]$ of $i_{0,\beta}(I_{\mathcal{E}})$ which is satisfied by x in the sense that $x \in A(\varphi)$. Apply this extender to the appropriate model on the tree (according to the requirements in the definition of iteration tree). At limit stages pick the unique cofinal well-founded branch if it exists. Suppose case 1. does not obtain and the construction lasts κ stages producing a tree T of length κ with cofinal well-founded branch b and an embedding $i_b : M \longrightarrow M_b$. For a model M_β on the tree let M^*_β denote $i_{0,\beta}(V^M_{\bar{\delta}})$. Thus T may be viewed as a tree on these smaller structures. Inside $M[g]$ let $j : H \longrightarrow V_\xi$, where ξ is large enough, H is countable and transitive and j is elementary with $x, b, T \in \operatorname{ran}(j)$. Let $\alpha = H \cap \kappa$. It is easy to see that $\alpha \in b$, $j^{-1}(M^*_b) = M^*_\alpha$ and

$$j \restriction M^*_\alpha = i_{\alpha,b} \restriction M^*_\alpha$$

giving the usual contradiction.

We now show that cases 1. and 2. both contradict our original smallness assumption. The second case leads to a contradiction as follows. Let $i^* : M \longrightarrow M^*$ which we have assumed is definable in $M[g]$. By assumption, x is generic for $i^*(W_{\bar{\delta}}(\mathcal{E}))$. Clearly $M^*[x] = M[g]$. By the chain condition $i^*(\bar{\delta})$ is a regular cardinal of $M^*[x]$ and hence is equal to κ. This is a contradiction because $M[g]$ sees that $i^*(\bar{\delta})$ is a countable ordinal. So we may assume that the first case obtains. Let λ be the limit length of the tree T for which $M[g]$ does not see a well-founded cofinal branch. In the outside world, there is a unique cofinal well-founded branch b_λ with final well-founded model M_λ.

Claim 2.2 *For any $\gamma < \eta$ there is $h \subset \operatorname{Col}(\omega, \gamma)$ which is $M[g]$-generic such that $M[g][h]$ sees distinct cofinal branches of T with γ in the well-founded part of both final models.*

Proof. Let $\gamma < \eta$. In any such $M[g][h]$ there is a cofinal branch b of T with γ in the well-founded part of M_b by Claim 2.1 as the sentence asserting the existence of such a branch is Σ^1_2 in any code for a well-ordering of length γ. If this branch were unique, then necessarily $b = b_\lambda$ and we would have $b_\lambda \in M[g]$ by homogeneity of the forcing $\operatorname{Col}(\omega, \gamma)$, contrary to our assumption. □

Thus in any such $M[g][h]$ the model $L_\eta(M(T))$ thinks that $\delta(T)$ is Woodin by Theorem 1.5 and hence $M[g]$ sees that $L_\eta(M(T))$ thinks that $\delta(T)$ is Woodin. Let $i_\lambda : M \longrightarrow M_{b_\lambda}$ be the branch embedding, which exists outside of M. Thus M_{b_λ} thinks that some ordinal $\alpha \leq i_\lambda(\bar{\delta})$ is Woodin in $L(V_\alpha)$ so M thinks the same of some ordinal $\alpha \leq \bar{\delta}$ contradicting our assumption. An alternate argument suggested by the referee is that since $M[g]$ sees T and $M(T)$ and the map i_λ is elementary, one has that $L(M(T))$ thinks that $\delta(T)$ is not Woodin. Hence by the proof of Claim 2.1, $L_\eta(M(T))$ also sees that $\delta(T)$ is not Woodin and the Q-structure $Q(b_\lambda, T)$ (see [3]) of M_{b_λ} belongs to $M[g]$ from which it follows b_λ must belong to $M[g]$ contradicting our assumption.

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