PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 5, May 2008, Pages 1807–1814
S 0002-9939(08)08918-1
Article electronically published on January 17, 2008

ON A FRAGMENT OF THE UNIVERSAL BAIRE PROPERTY FOR Σ_2^1 SETS

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(Communicated by Julia Knight)

ABSTRACT. There is a well-known global equivalence between Σ_2^1 sets having the universal Baire property, two-step Σ_3^1 generic absoluteness, and the closure of the universe under the sharp operation. In this note, we determine the exact consistency strength of Σ_2^1 sets being $(2^\omega)^+$-cc-universally Baire, which is below $0^\#$. In a model obtained, there is a Σ_2^1 set which is weakly ω_2-universally Baire but not ω_2-universally Baire.

1. Introduction

Consider the following two properties of a set of reals $A \subset \omega^\omega$ at an infinite cardinal κ:

(1) For every continuous $f : \kappa^\omega \to \omega^\omega$, there is a dense set of $p \in \kappa^{<\omega}$ such that $f^{-1}(A)$ is either meager or comeager below p.
(2) For every continuous $f : \kappa^\omega \to \omega^\omega$, there is a dense set of $p \in \kappa^{<\omega}$ such that $f^{-1}(A) \cap \sigma^\omega$ is either meager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$ or comeager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$.

The first property asserts that A is κ-universally Baire or fully captured at κ, and the second asserts that A is weakly κ-universally Baire (the author's coinage) or weakly captured at κ (see [9] and Lemma 4.1 below). The implication of (1) to (2) is immediate as a club of $\sigma \in [\kappa]^\omega$ is closed under a Banach–Mazur strategy in the space κ^ω. Regarding the reverse implication, any set of reals of size ω_1 is a counterexample at $\kappa = \omega_2$ assuming Martin's Maximum (see Theorem 2.6 of [9] and Theorem 3.1 of [7]). The place to look for a definable counterexample is the pointclass Σ_2^1 with κ either ω_1 or ω_2. This is because (1) and (2) are equivalent for $\kappa = \omega$ and for Δ_2^1 sets as a whole. Since the particular scenario for a counterexample suggested by [9] involves the question of the consistency strength of Σ_2^1 sets being ω_1- or ω_2-universally Baire, in particular whether this is possible without sharps, this paper was motivated by the following question: What fragment of the universal Baire property can Σ_2^1 sets have below $0^\#$, as measured by the weight or cellularity of the preimage space? In Theorem 3.4 of [1] a global equivalence is established between Σ_2^1 sets being universally Baire, two-step Σ_3^1 generic absoluteness, and the closure of the universe under the sharp operation. This equivalence, however, is not true level-by-level. In particular, the relevant part of their argument (originating

Received by the editors March 20, 2006 and, in revised form, September 12, 2006.
2000 *Mathematics Subject Classification.* Primary 03E45; Secondary 03E35.
Key words and phrases. Generic absoluteness, universal Baire property.

in [8]) would require that Σ^1_2 sets be $\omega_{\omega+1}$-universally Baire to prove the existence of $0^\#$. Using the full strength of covering for L, this can be reduced to ω_3. On the other hand, Woodin has shown that Π^1_2 sets can be ω_2-cc-universally Baire in a forcing extension of L.

Definition 1.1. $A \subset \mathbb{R}$ is κ-cc-universally Baire if $f^{-1}(A)$ has the Baire property in X for every completely regular space X of cellularity less than κ and every continuous map $f : X \to \mathbb{R}$.

Theorem 1.2 (Woodin). *Assume $\lambda_0 < \lambda_1 < \lambda_2$ are cardinals of L and that there is an elementary embedding $\pi : L_{\lambda_1} \to L_{\lambda_2}$ with a critical point λ_0 such that $\pi(\lambda_0) = \lambda_1$. Then Σ^1_2 sets are ω_2-cc-universally Baire in a forcing extension of L in which CH holds.*

For equivalent versions of Definition 1.1, the reader is referred to Theorem 2.1 of [1]. In particular, we will use that a κ-cc-universally Baire set remains ccc-universally Baire after forcing with a κ-cc poset. When combined with the argument of Theorem 3.4 of [1], the above shows that two-step Σ^1_3 generic absoluteness for $(2^\omega)^+$-cc forcings $\mathbb{P} * \dot{\mathbb{Q}}$ can hold in a forcing extension of L. In this note we reduce the hypothesis of Theorem 1.2 to obtain an exact equiconsistency.

Definition 1.3. An ordinal κ is L-large to λ if for every $\alpha < \lambda$ there is an elementary $j : L_\alpha \to L_\beta$ with critical point κ such that $j(\kappa) \geq \alpha$.

Note that an ordinal κ is L-large to $(\kappa^+)^L$ if and only if κ is weakly compact in L. We obtain stronger notions by requiring that λ be inaccessible or weakly compact in L. Neither notion implies that $0^\#$ exists (simply collapse λ^+ and use absoluteness of $L[g]$), though L cannot see such an embedding with $\alpha \geq (\kappa^+)^L$.

Theorem 1.4. *The following are equiconsistent:*

(1) *Σ^1_2 sets are $(2^\omega)^+$-cc-universally Baire;*
(2) *There is a κ which is L-large to a weakly compact L.*

In the last section of this paper, we argue that in the model of Theorem 1.4(1) there must be a Σ^1_2 set which has a weak capturing term at ω_2 but no full capturing term at ω_2.

Theorem 1.5. *It is consistent that Σ^1_2 sets are weakly ω_2-universally Baire but not ω_2-universally Baire.*

In what follows, any space X^ω will carry the product topology, with the set X given the discrete topology. All pointclasses are boldface, and every statement below involving Σ^1_2 sets applies equally well to Π^1_2 sets. We would like to thank Hugh Woodin for sharing his proof of Theorem 1.2 and allowing us to include elements of it here and Stevo Todorcevic for several helpful comments regarding earlier drafts of this paper.

2. Preliminaries

Call a term $\dot{A}$ a $Col(\omega, \kappa)$ capturing term for a set of reals A if there is a club of countable elementary submodels $X \prec H(\theta)$ with transitivization H and collapse map π such that

$$\pi(\dot{A})_g = A \cap H[g]$$

for every H-generic $g \subset Col(\omega, \pi(\kappa))$. A set of reals A has a $Col(\omega, \kappa)$ capturing term if and only if A is κ-universally Baire (see Lemma 1.6 of [9]). The proof below uses this observation and an argument from [1].

Theorem 2.1. *The following are equivalent for a cardinal κ:*

(1) *Σ^1_2 sets are κ-universally Baire;*
(2) *For all sufficiently large θ, there is a club of countable $X \prec H(\theta)$ such that $X[g]$ is Σ^1_2 elementary in V for every X-generic $g \subset Col(\omega, \kappa \cap X)$ which belongs to V.*

Proof. Let A be Σ^1_2 defined by a formula ϕ (we suppress any parameter). Assuming (2), let $\dot{A}$ be the set of pairs (p, τ) such that $p \Vdash \phi(\tau)$. Thus $\dot{A}$ is a capturing term for A and it follows that A is κ-universally Baire. For the other direction assume S and T are trees which witness that a given Σ^1_2 set is κ-universally Baire. Suppose this Σ^1_2 set is defined by a formula $\phi(x)$ (again suppressing parameters). The argument of Theorem 3.4 of [1] shows that

$$p[S]^{V[G]} = \{x \mid \phi(x)\}^{V[G]},$$

where $G \subset Col(\omega, \kappa)$ is V-generic. This uses Π^1_1-uniformization. Let ϕ be a Σ^1_2 formula defining the universal Σ^1_2 set A and let $X \prec H(\theta)$ contain S and T. Then $p[S]^{X[g]} = A \cap X[g]$ and $X[g]$ thinks $p[S]^{X[g]}$ is the universal Σ^1_2 set. Hence $X[g]$ is Σ^1_2 elementary in V. □

Using (2) and the full strength of covering for L, we may now argue that Σ^1_2 sets being ω_3-universally Baire imply that $0^\#$ exists. For a set of ordinals σ we let $otp(\sigma)$ denote the order type of σ.

Theorem 2.2. *Assume Σ^1_2 sets are ω_3-universally Baire. Then $0^\#$ exists.*

Proof. Let $\kappa = \omega_3$. We first argue that there are club many $\sigma \in [\kappa]^\omega$ such that $otp(\sigma)$ is a regular cardinal of L. Let $\kappa \in X \prec H(\theta)$ be countable with transitive collapse $\pi : X \to \bar{X}$. Note that $\pi(\kappa) = otp(X \cap \kappa)$ and that there are club many such $X \cap \kappa$. Thus $\bar{X}$ thinks that $\pi(\kappa)$ is a cardinal of L. If $\pi(\kappa)$ were not a regular cardinal of L, then there would be a countable L_γ which sees this. Let $g \subset Col(\omega, \pi(\kappa))$ be $\bar{X}$-generic, and let $z \in \bar{X}[g]$ be a real coding a well-ordering of length $\pi(\kappa)$. Then by Theorem 2.1, $\bar{X}[g]$ thinks there is a level of L which sees that the ordinal coded by z is not regular. This is a contradiction as $L^{\bar{X}} = L^{\bar{X}[g]}$. We now argue that the set of $\alpha < \kappa$, such that α is a regular cardinal of L, contains a club in V. Let $f : \kappa^{<\omega} \to \kappa$ be such that any $\sigma \in [\kappa]^\omega$ which is closed under f has the property that $otp(\sigma)$ is a regular cardinal of L. Let $\alpha < \kappa$ such that $f[\alpha^{<\omega}] \subset \alpha$. Since there are club many such α, it suffices to show that α is a regular cardinal of L. Suppose not. There is a countable $X \prec H(\kappa)$ with $\alpha \in X$ such that $X \cap \alpha = \sigma$ is closed under f. Let $\bar{X}$ be the transitivization of X with a collapse map π. Then $\bar{X}$ thinks that $\pi(\alpha)$ is not a regular cardinal of L; hence, $\pi(\alpha)$ is not a regular cardinal of L by absoluteness. This contradicts $\pi(\alpha) = otp(\sigma)$. It follows that there is an α with $cf(\alpha) < \omega_2 < \alpha < \omega_3$ which is a regular cardinal of L. Let $\sigma \subset \alpha$ be unbounded in α and have size $cf(\alpha)$. Then σ cannot be covered by a set in L of size ω_1. □

We conjecture that ω_3-cc-universally Baire suffices for the argument above. Under this assumption ω_3 is weakly compact in L by Lemma 4 of [8] and a theorem in

[2]. We close this section with an equivalence between Σ^1_2 sets being ω_1-universally Baire and the existence of a club of suitably closed submodels. We say that ω_2 is inaccessible to $P(\omega_1)$ if ω_2 is an inaccessible cardinal in $L[X]$ for every $X \subseteq \omega_1$.

Lemma 2.3. *The following are equivalent:*

(1) *ω_2 is inaccessible to $P(\omega_1)$ and Σ^1_2 sets are ω_1-universally Baire;*
(2) *ω_2 is inaccessible to $P(\omega_1)$ and for sufficiently large θ there is a club of $X \prec H(\theta)$ such that for every $\tau \in P(\omega_1) \cap X$ and every $L[\tau]$-cardinal $\gamma \in X \cap \omega_2$ the order type of $\gamma \cap X$ is itself an $L[\tau \cap X]$-cardinal;*
(3) *For sufficiently large θ there is a club of $X \prec H(\theta)$ such that for every $\tau \in P(\omega_1) \cap X$ the order type of $X \cap \omega_2$ is an $L[\tau \cap X]$-cardinal.*

Proof. By the argument of Theorem 2.2, condition (3) implies that ω_2 is inaccessible to $P(\omega_1)$. Thus (2) and (3) are equivalent. Again by a boldface version of an argument from Theorem 2.2, (1) implies (2). Let $X \prec H(\theta)$ be as in (2). Let $\pi : X \to \bar{X}$ be the collapse map and let $g \subset Col(\omega, \omega_1 \cap X)$ be $\bar{X}$-generic. Let $y = \pi(\tau)_g$ be a real in $\bar{X}[g]$. Since g is also $L[\pi(\tau)]$-generic, we have

$$\pi(\omega_2) > (\pi(\omega_1)^+)^{L[\pi(\tau)]} \geq (\omega_1)^{L[y]}$$

so that $\bar{X}[g]$ is correct about Σ^1_2 facts in the parameter y. $\square$

3. Equiconsistency results

Fix a surjection $f_\gamma : \omega_1 \to \gamma$ for each γ between ω_1 and ω_2. If γ is a cardinal of L, let S_γ denote the set of $\alpha < \omega_1$ such that the order type of $f_\gamma[\alpha]$ is an L-cardinal. Let S be the set of $\sigma \in [\omega_2]^\omega$ such that

$$\sigma \cap \omega_1 \in \bigcap_{\gamma \in \sigma} S_\gamma.$$

If we assume that ω_2 is inaccessible in L and that there are stationary many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is an L-cardinal, then it follows that S is stationary. Now let $\mathbb{Q}$ be the countable support product of $\mathbb{Q}_\gamma$, ranging over ordinals $\gamma < \omega_2$ which are L-cardinals, where $\mathbb{Q}_\gamma$ is the poset for shooting a club through S_γ with countable conditions. It follows that $\mathbb{Q}$ is (ω, ∞)-distributive. If CH holds, then $\mathbb{Q}$ satisfies the ω_2-chain condition. The following key lemma is implicit in Woodin's proof of Theorem 1.2. We thank the referee for pointing out that condition (2) below is considered in [3].

Lemma 3.1. *Suppose that:*

(1) *every subset of ω_1 is L-generic for some poset $\mathbb{P} \in L$ with $|\mathbb{P}| < \omega_2$;*
(2) *there are stationary many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is an L-cardinal;*
(3) *ω_2 is inaccessible in L and CH holds.*

Then Σ^1_2 sets are ω_1-universally Baire in $V[G]$ where $G \subset \mathbb{Q}$ is V-generic.

Proof. We show that condition (3) of Lemma 2.3 is satisfied in $V[G]$. As discussed above, $\mathbb{Q}$ preserves cardinals under these hypotheses and by design there is in $V[G]$ a club of $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is a cardinal of L. Furthermore, condition (1) continues to hold in $V[G]$. Suppose $X \prec H(\theta)$ is such that $otp(X \cap \omega_2)$ is an L-cardinal. Let $\tau \in P(\omega_1) \cap X$. Then there are $\mathbb{P}, H \in X$ such that X thinks that $\mathbb{P} \in L$, $|\mathbb{P}| < \omega_2$, $H \subset \mathbb{P}$ is L-generic and $\tau \in L[H]$. Let $\pi : X \to \bar{X}$ be the transitivization map. As $otp(X \cap \omega_2) = (\omega_2)^{\bar{X}}$ is a limit cardinal of L, it follows that $\pi(H) \subset \pi(\mathbb{P})$ is L-generic and $\tau \cap X \in L[\pi(H)]$. Thus $otp(X \cap \omega_2)$ remains a cardinal in $L[\tau \cap X]$ as desired. □

Schindler pointed out to the author that condition (3) below is equiconsistent with the existence of a cardinal which is remarkable up to an inaccessible cardinal, a notion from his papers [5] and [6]. This large cardinal concept has the advantage of not mentioning an inner model in its definition. We include this observation without proof.

Theorem 3.2. *The following are equiconsistent:*

(1) *ω_2 is inaccessible in L and Σ^1_2 sets are ω_1-universally Baire;*
(2) *There are club many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is an L-cardinal;*
(3) *ω_2 is inaccessible in L and there are stationary many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is a cardinal of L;*
(4) *There is a κ which is L-large to an L-inaccessible;*
(5) *There is a cardinal κ which is remarkable up to an inaccessible cardinal.*

Proof. (1) implies (2) outright by Lemma 2.3. The argument for (2) implies (3) is implicit in the proof of Theorem 2.2. Assume (3). Let $g \subset Col(\omega, < \omega_1)$ be V-generic. Let $\kappa = \omega_2^V$. Then in $L[g]$ there is a stationary set of $\sigma \in [\kappa]^\omega$ such that the order type of σ is an L-cardinal. Thus if $h \subset Col(\omega_1, < \omega_2^V)$ is $L[g]$-generic in $L[g][h]$, then the hypotheses of Lemma 3.1 are satisfied so that (1) holds in the forcing extension described there. Thus (1), (2) and (3) are equiconsistent. Assume (1). We will show that ω_1^V is L-large to ω_2^V in $V[g]$ where $g \subset Col(\omega, \omega_1)$ is V-generic. Let $X \prec H(\theta)$ be countable. Let $\pi : X \to H$ be the transitive collapse. Let $Y \prec H(\theta)$ with $\pi, H \in Y$ and let $j : Y \to M$ be its transitivization. Note that $j \circ \pi^{-1} = j(\pi)$. Call this map k. We have that

$$k \upharpoonright L_{\omega_2^H} : L_{\omega_2^H} \to L_\gamma$$

is fully elementary with critical point ω_1^H and this map is an element of M. Because Y sees that H is countable, we have

$$k(\omega_1) = Y \cap \omega_1 > \omega_2^H.$$

Let $g \subset Col(\omega, \omega_1^H)$ be H-generic. Let $\alpha < \omega_2^H$ be arbitrary and let $x \in H[g]$ be a real coding a well-ordering of length α. The sentence asserting the existence of a transitive model of a sufficient fragment of set theory containing y which sees an embedding $k : L_\alpha \to L_\beta$ with critical point ω_1^H such that $j(\omega_1^H) > \alpha$ is Σ^1_2 in the parameter x. Hence $H[g]$ sees such an embedding. As α is arbitrary, we conclude that $H[g]$ thinks that ω_1^H is L-large to ω_2^H. Now apply π. To connect (4) back to (1), assume that κ is L-large to some L-inaccessible λ. Let $g_\lambda \subset Col(\omega, < \lambda)$ be V-generic. Then g_λ is also L-generic for the same forcing. By folding the embeddings witnessing our hypothesis (4) into countable submodels and collapsing, we see that κ is L-large to $\lambda = \omega_1^{L[g_\lambda]}$ in $L[g_\lambda]$. For ordinals $\gamma < \lambda$ let g_γ denote $g \cap Col(\omega, < \gamma)$. We claim that in $L[g_\kappa]$, there are stationary many $\sigma \in [\lambda]^\omega$ such that $otp(\sigma)$ is an L-cardinal. It will then follow that (3) holds after forcing with $Col(\omega_1, < \lambda)$. Let $f : \lambda^{<\omega} \to \lambda$ belonging to $L[g_\kappa]$ be arbitrary. Let $\delta < \lambda$ be a cardinal of $L[g_\kappa]$ such that $f[\delta^{<\omega}] \subseteq \delta$. Let α be a regular cardinal of $L[g_\kappa]$ below λ which is greater than $(\delta^+)^L$ so that $f \upharpoonright \delta^{<\omega} \in L_\alpha[g_\kappa]$. In $L[g_\lambda]$ there is an elementary embedding

$$j : L_\alpha \to L_\beta$$

with a critical point κ such that $j(\kappa) > \alpha$. By standard arguments (using the fact that $Col(\omega, < \kappa)$ is κ-cc in L) this embedding extends to a fully elementary

$$j : L_\alpha[g_\kappa] \to L_\beta[g_{j(\kappa)}].$$

The embedding is defined by $j(\text{val}(\tau, g_\kappa)) = \text{val}(j(\tau), g_{j(\kappa)})$ and since it extends j, we will also denote it by j. Now, let σ denote the set $j[\delta]$. Let z be a real in $L_\beta[g_{j(\kappa)}]$ coding a well-ordering of length δ. The structure $L_\beta[\hat{g}]$ has a tree T consisting of pairs (s, t) with s a finite approximation to a set of ordinals σ closed under f and with t a finite approximation to an order isomorphism between δ and σ. Moreover, $L_\beta[g_{j(\kappa)}]$ must see a branch through this tree and the result follows by reflection using j. □

Lemma 3.3. *Suppose κ is a weakly compact cardinal and $\mathbb{P}$ is κ-cc. Suppose Π^1_2 sets are $< \kappa$-universally Baire in $V[G]$ where $G \subset \mathbb{P}$ is V-generic. Then Π^1_2 sets are κ-cc-universally Baire in $V[G]$.*

Proof. We will use the fact that a set $A \subset \omega^\omega$ is κ-cc-universally Baire iff there are trees S, T on some $\omega \times \lambda$ which project to A and its complement and continue to project to complements after forcing with any κ-cc poset. So let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset which is forced by $\mathbb{P}$ to have the κ-cc. Thus $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-cc in V. Now suppose $\dot{x}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a real. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc and κ is weakly compact, there is an elementary suborder $\mathbb{A} \prec \mathbb{P} * \dot{\mathbb{Q}}$ which has size strictly less than κ, decides $\dot{x}$, and has the property that maximal $\mathbb{A}$ antichains are maximal antichains in $\mathbb{P} * \dot{\mathbb{Q}}$. The upshot of this is that over $V[G]$ where $G \subset \mathbb{P}$ is V-generic, every real which is generic for a κ-cc forcing is generic for a forcing of size $< \kappa$. Let A be a Σ^1_2 set and for each forcing $\mathbb{Q}$ of size $< \kappa$ (whose underlying set is some ordinal below κ say) let $S_{\mathbb{Q}}, T_{\mathbb{Q}}$ be $\mathbb{Q}$-universally Baire representations of A. These trees may be joined to produce the desired κ-cc-universally Baire representation of A. □

Schindler pointed out to the author that condition (4) could be added to the theorem below (see the remarks preceding Theorem 3.2).

Theorem 3.4. *The following are equiconsistent:*

1. *Σ^1_2 sets are $(2^\omega)^+$-cc-universally Baire;*
2. *ω_2 is weakly compact in L and there are stationary many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is a cardinal of L;*
3. *There is a κ which is L-large to a weakly compact cardinal of L;*
4. *There is a cardinal which is remarkable up to a weakly compact cardinal.*

Proof. This is identical to the proof of Theorem 3.2, using Theorem 3.4 in the argument from (2) to (1) to get the stronger conclusion. Of course we are using that CH holds in all models under consideration. We need to show that (1) implies that ω_2 is weakly compact in L. Assume (1) and let $\mathbb{Q}$ be the poset for forcing Martin's Axiom. Let $\mathbb{P} = Col(\omega, \omega_1) * \mathbb{Q}$ and note that $\mathbb{P}$ is ω_2-cc. In the extension $V[G]$ by $\mathbb{P}$ we will have Σ^1_2 sets ccc-universally Baire. This implies that ω_1 is inaccessible to reals in this model by a result in [8]. Thus $\omega_1 = \omega_2^V$ is weakly compact in L in $V[G]$ by a result of Harrington and Shelah (see [2] or Lemma 7 of [4]). □

4. Weak capturing does not imply capturing

A weakening of the universal Baire property is presented in [6]. A set of reals A is weakly captured at κ if there is a $Col(\omega,\kappa)$-term $\dot{A}$ such that for sufficiently large θ, for a club of countable $H \prec H(\theta)$, and for a comeager set of $g : \omega \to otp(H \cap \kappa)$,

$$\pi_H(\dot{A})_g = A \cap H[g],$$

where $otp(H \cap \kappa)$ is the order type of $H \cap \kappa$ and π_H is the transitivization map. A less metamathematical characterization is the following.

Lemma 4.1. *The following are equivalent:*

(1) *A is weakly captured at κ;*
(2) *For every continuous $f : \kappa^\omega \to \omega^\omega$, there is a dense set of $p \in \kappa^{<\omega}$ such that $f^{-1}(A) \cap \sigma^\omega$ is either meager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$ or comeager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$.*

Proof. (1) implies (2) is immediate as any condition $p \in \kappa^{<\omega}$ has a refinement $p \subseteq q$ such that $q \Vdash_{Col(\omega,\kappa)} f(\dot{g}) \in \dot{A}$ or $q \Vdash_{Col(\omega,\kappa)} f(\dot{g}) \notin \dot{A}$. For the other direction, if τ is a standard $Col(\omega,\kappa)$ term for a real, then τ gives rise to a function $f_\tau : \kappa^\omega \to \omega^\omega$ defined by $f_\tau(g) = \tau_g$ which is continuous on a comeager set. Define $\dot{A}$ to be the set of (p,τ) such that $f^{-1}(A) \cap \sigma^\omega$ is comeager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$. A straightforward argument shows that $\dot{A}$ is a weak capturing term for A. □

Theorem 4.2. *It is consistent relative to the existence of a cardinal which is remarkable up to a weakly compact cardinal that Σ^1_2 is weakly captured at ω_2 but not fully captured at ω_2.*

Proof. Suppose κ is L-large to an L-weakly compact, and let $L[g][h]$ be the model of Theorem 3.2 in which $\kappa = \omega_1$ and $\lambda = \omega_2$. We have shown that there is a stationary set $S \subset [\omega_2]^\omega$ such that whenever $X \prec H(\theta)$ is such that $X \cap \omega_2 \in S$, then $X[g]$ is Σ^1_2 elementary in V for every X-generic $g \subset Col(\omega, \omega_1 \cap X)$. The forcing $\mathbb{Q}$ of Lemma 3.1 puts a club through S and so in the extension all Σ^1_2 sets are ω_1-universally Baire. We first argue that $WRP_{(2)}(\omega_2)$ holds in $L[g][h][G]$. For $a \subseteq \lambda$, let $\mathbb{Q}_a$ denote the countable support product of $\mathbb{P}_\gamma$ taken over L-cardinals $\gamma \in a$, with S the underlying stationary set. Returning to $L[g]$ where λ is still weakly compact, let p be a condition and $\dot{T}$ a term such that

$$p \Vdash^{L[g]}_{Col(\omega_1,<\lambda)*\mathbb{Q}_\lambda} \dot{T} \text{ is stationary in} [\omega_2]^\omega.$$

By the usual reflection argument we have an inaccessible $\delta < \omega_2$ such that

$$p \Vdash^{L[g]}_{Col(\omega_1,<\delta)*\mathbb{Q}_\delta} \dot{T}_\delta \text{ is stationary in } [\delta]^\omega,$$

where $\dot{T}_\delta$ denotes $\dot{T} \cap V_\delta$. Now let $h \subset Col(\omega_1, <\lambda)$ be $L[g]$-generic and let $G \subset \mathbb{Q}$ be $L[g][h]$-generic below the condition p. Let h_δ and G_δ be the restrictions to $Col(\omega_1, <\delta)$ and $\mathbb{Q}_\delta$, respectively. These are $L[g]$-generic as well and

$$val(\dot{T}_\delta, h_\delta * G_\delta) = val(\dot{T}, h * G) \cap [\delta]^\omega$$

in $L[g][h][G]$. We need to show that the stationarity of $T_\delta = val(\dot{T}_\delta, h_\delta * G_\delta)$ is preserved. It suffices to show that the stationarity of T is preserved by $\mathbb{Q}_{\lambda\setminus\delta}$ over $L[h][G_\delta]$. The key point is that $\{\sigma \cap \delta \mid \sigma \in S\}$ contains a club in $[\delta]^\omega$. Thus if $\dot{C}$ is a name for a club subset of $[\delta]^\omega$, we can find a dense set of conditions $t \in \mathbb{Q}_{\lambda\setminus\delta}$

with a corresponding $\sigma \in S$ such that $\sigma \cap \delta \in T$ and t forces $\sigma \in \dot{C}$. Now let G be $L[g][h]$-generic for the forcing $\mathbb{Q}$. This forcing does not add countable sets of ordinals. Let A be a Σ^1_2 set in $L[g][h][G]$. Then $A = A^{L[g][h]}$. Fix such an A. We claim that A is weakly captured in $L[g][h][G]$. Otherwise, there is a condition $t \in \mathbb{Q}$, terms $\dot{T}_m$ and $\dot{T}_c$, and $p \in \omega_2^{<\omega}$ such that t forces the following to hold:

(1) $\dot{T}_m$ and $\dot{T}_c$ are both stationary subset of S;
(2) $\sigma \in \dot{T}_m$ implies $\dot{f}^{-1}(A) \cap \sigma^\omega$ is meager below p;
(3) $\sigma \in \dot{T}_c$ implies $\dot{f}^{-1}(A) \cap \sigma^\omega$ is comeager below p.

We may assume that there is a $\delta < \omega_2$ such that t forces both $\dot{T}_m$ and $\dot{T}_c$ to reflect to δ. Let $\bar{t} \leq t$ and $\bar{p} \leq p$ such that

$$\bar{t} \Vdash_{\mathbb{Q}} \dot{f}^{-1}(A) \cap \delta^\omega \text{ is comeager below } \bar{p}.$$

It follows that $\bar{t}$ forces that $\dot{f}^{-1}(A) \cap \sigma^\omega$ is comeager below $\bar{p}$ for a club of $\sigma \in [\delta]^\omega$, a contradiction. To finish the proof of the theorem, we must show that Σ^1_2 sets are not ω_2-universally Baire in $L[g][h][G]$. Let D be the set of $\alpha < \lambda$ such that $cf(\alpha) = \omega$ in L. As $L[g][h][G]$ is a λ-cc extension of L, we know that D remains stationary in $L[g][h][G]$. Thus the set of regular cardinals of L below ω_2 cannot be club, as they be would be if Σ^1_2 sets were ω_2-universally Baire by the argument of Theorem 2.2. □

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