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Topological group criterion for $C(X)$ in compact-open-like topologies, I

R. Ball [a], V. Gochev [b,*], A. Hager [c], S. Todorčević [d], S. Zoble [c]

[a] Department of Mathematics, University of Denver, Denver, USA
[b] Department of Mathematics, Trinity College, Hartford, USA
[c] Department of Mathematics and Computer Science, Wesleyan University, Middletown, USA
[d] Department of Mathematics, University of Toronto, Toronto, Canada

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ABSTRACT

We address questions of when $(C(X),+)$ is a topological group in some topologies which are meets of systems of compact-open topologies from certain dense subsets of X. These topologies have arisen from the theory of epimorphisms in lattice-ordered groups (in this context called "epi-topology"). A basic necessary and sufficient condition is developed, which at least yields enough insight to provide the general answer "sometimes Yes and sometimes No". After some reduction the situation seems to become Set Theory (which view will be reinforced by a sequel to this paper "Topological group criterion for $C(X)$ in compact-open-like topologies, II").

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1. Introduction

While the motivation for the problems discussed in this paper lie in the theory of lattice-ordered groups, description of this connection is confined here to an Appendix. The rest of the paper is, in its technical details, independent of that theory.

* Corresponding author.
E-mail address: Vasil.Gochev@trincoll.edu (V. Gochev).

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In the discussion immediately below, we define the topologies at issue, and state our main results, Theorems 1.3 and 1.4. Section 2 comes to technical grips with these topologies, and in Sections 3 and 4, we prove these main results. Section 5 is the Appendix mentioned above.

Consider compact Hausdorff X. The set $C(X)$ of real-valued continuous functions on X, with pointwise addition $((f+g)(x) \equiv f(x)+g(x))$, is an abelian group (and much more, of course, e.g., [11,19]).

Consider a filter base $\mathfrak{F}$ of dense cozero-sets in X (equivalently, dense open F_σ's). Let $\mathfrak{F}_\delta = \{\bigcap \mathfrak{F}' \mid \text{countable } \mathfrak{F}' \subseteq \mathfrak{F}\}$. For each $F \in \mathfrak{F}_\delta$, F is dense in X (Baire Category Theorem) and $C(X)$ is viewed as a subset of $C(F)$ via the injection $C(X) \ni f \mapsto f|F \in C(F)$. Let τ_F (respectively, σ_F) denote the compact-open (respectively, "compact-zero") topology of $C(F)$, traced on $C(X)$. (Here, "zero" means: use $\epsilon = 0$ in the definition of neighborhoods. See Section 2 below.) These are Hausdorff group topologies on $C(X)$.

The topologies considered in this paper are $\tau^{\mathfrak{F}} \equiv \bigwedge\{\tau_F \mid F \in \mathfrak{F}_\delta\}$ and $\sigma^{\mathfrak{F}} \equiv \bigwedge\{\sigma_F \mid F \in \mathfrak{F}_\delta\}$, these meets in the lattice of topologies on $C(X)$. These have arisen as tools to describe epimorphisms in the category of archimedean lattice-ordered groups with unit, and more generally, monomorphisms in the category of spaces with Lindelöf filter (an object of which is exactly an $(X, \mathfrak{F})$ as above). The connections come from [2,3,12], and are described in the Appendix.

These topologies are always T_1, homogeneous, inversion is continuous, and $+$ is separately continuous. We comment briefly on the Hausdorff property in Section 2 (sometimes Yes and sometimes No). These topologies are also countably tight [2,13], but we know hardly anything about further features of $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$. See [1] and [18] for a wealth of inspiration on particular questions one might ask.

This paper is a first address (after minor remarks in [2] anyway) to the question: When is $+$ jointly continuous (i.e., $C(X)$ is a topological group)? (A second "address" will be a sequel "Topological group criterion for $C(X)$ in compact-open-like topologies, II" to this paper, described at the end of this section.) It is immediate that

Proposition 1.1. *If $\mathfrak{F}_\delta$ has a minimum element F_0, then $\tau^{\mathfrak{F}} = \tau_{F_0}$ and $\sigma^{\mathfrak{F}} = \sigma_{F_0}$, so these are group topologies on $C(X)$.*

While examples of the basic situation $(X, \mathfrak{F})$ are myriad, our favorite examples (and the only ones seriously considered in this paper) have the form $(\beta Y, \mathcal{C})$, where Y is a Tychonoff space, βY is the Čech–Stone compactification, and $\mathcal{C}$ denotes all cozero-sets in βY which contain Y. These are "favorite" because here the $\tau^{\mathcal{C}}$ is the "epi-topology" on $C(Y)$ from [2] traced on the bounded functions $C^*(Y) = C(\beta Y)$, and the epi-topology is a group topology iff $\tau^{\mathcal{C}}$ is. For $\sigma^{\mathcal{C}}$, the situation is similar [12]. (See the Appendix.) Then $\tau^{\mathcal{C}}$ and $\sigma^{\mathcal{C}}$ are always Hausdorff. (See Section 2 below.) Now, a space is called Čech-complete if it is G_δ in its Čech–Stone compactification, or in every compactification [10]. Thus, for any $(X, \mathfrak{F})$, the members of $\mathfrak{F}_\delta$ are Čech-complete, and also Lindelöf [10, p. 201]. Alluding to Proposition 1.1 for the favorite situation $(\beta Y, \mathcal{C})$, $\mathcal{C}_\delta$ has a minimum element iff the Hewitt realcompactification υY is Lindelöf and Čech-complete (since always $\upsilon Y = \bigcap \mathcal{C} = \bigcap \mathcal{C}_\delta$ [11]). Thus

Corollary 1.2. *If υY is Lindelöf and Čech-complete, then $\tau^{\mathcal{C}}$ and $\sigma^{\mathcal{C}}$ are group topologies on $C(\beta Y)$.*

In Theorem 2.5 below, we give a necessary and sufficient condition that a general $\tau^{\mathfrak{F}}$ (and $\sigma^{\mathfrak{F}}$) be a group topology, and then apply this to produce the following two theorems/examples.

Let $D(\alpha)$ be the discrete space of power α, and let $\lambda D(\alpha)$ be $D(\alpha)$ with one point adjoined, whose neighborhoods have countable complement. As above, we consider $(\beta Y, \mathcal{C})$ for $Y = D(\omega_1)$ and $\lambda D(\omega_1)$.

Theorem 1.3. *(See Section 3 below.) On $C(\beta\lambda D(\omega_1))$, $\tau^{\mathcal{C}}$ and $\sigma^{\mathcal{C}}$ are group topologies.*

(This shows that the converse of Theorem 1.2 fails: $\lambda D(\omega_1)$ is Lindelöf, but not Čech-complete.)

Theorem 1.4. *(See Section 4 below.) On $C(\beta D(\omega_1))$, $\tau^{\mathcal{C}}$ and $\sigma^{\mathcal{C}}$ are not group topologies. So too for $C(\beta Y)$, for appropriate Y containing an appropriately embedded copy of $D(\omega_1)$.*

The proofs involve, first, somewhat complicated reduction of the criterion of Section 2 to issues of combinatorial set-theory, then resolution of those issues: for Theorem 1.3, using the club filter on ω_1 and the pressing-down lemma; for Theorem 1.4, using an Aronszajn tree. In each case, no axioms beyond **ZFC** are used.

Based on Corollary 1.2, Theorems 1.3 and 1.4 a primitive conjecture could be: On $C(\beta Y)$, $\tau^{\mathcal{C}}$ (or $\sigma^{\mathcal{C}}$) is a group topology iff υY is Lindelöf. We do not know if "υY Lindelöf" is necessary (see Theorem 4.7 for partial result), but it is not sufficient by our sequel to this paper: "Topological group criterion for $C(X)$ in compact-open-like topologies, II". On $C(\beta\mathbb{Q})$ ($\mathbb{Q}$ the rationals), under the Continuum Hypothesis, $\sigma^{\mathcal{C}}$ is not a group topology (and we do not know whether **CH** is needed, nor anything about $\tau^{\mathcal{C}}$); $\beta\mathbb{Q}$ is one among several similar examples. We do not include those results here because of the vagaries, the involvement of axioms beyond **ZFC**, and the significant extra complications and length.

2. The topologies, and the group criteria

This section (1) explains why $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$ are homogeneous topologies, so that neighborhoods at 0 suffice for most purposes, (2) describes convenient neighborhood bases at 0, (3) formulates the topological group property in terms of these bases. Sections 3 and 4 will analyze the group property in these terms, for the specific cases of Theorems 1.3 and 1.4.

The topological spaces variously denoted $X, Y, F, S, \ldots$ are Tychonoff (though not necessarily so for the various spaces of functions). For any space S, $\mathcal{K}(S)$ denotes the family of all compact subsets. We abbreviate "neighborhood" to "nbd".

On $C(S)$, the compact-open (*co*) and compact-zero (*cz*) topologies have, at each $f \in C(S)$, the local nbd bases respectively

(*co*) all sets $\{g \in C(S) \mid |f-g| \leqslant \epsilon \text{ on } K\}$ $(K \in \mathcal{K}(S),\ \epsilon \in (0,1))$,
(*cz*) all sets $\{g \in C(S) \mid f = g \text{ on } K\}$ $(K \in \mathcal{K}(S))$.

(These nbds are probably not open. A set is open iff it contains a nbd of each of its points.)

Each of these topologies on $C(S)$ is Hausdorff, makes $C(S)$ (with pointwise addition) a topological group, thus is homogeneous. (For *co*, this is well known: [7], [18], ... For *cz* the details are similar and described in [13].)

Let S be dense in X. The map $C(X) \ni f \mapsto f|S \in C(S)$ is a group embedding of $C(X)$ in $C(S)$, thus the *co* and *cz* topologies on $C(S)$, traced on $C(X)$ (i.e., the relative topology) are Hausdorff group topologies on $C(X)$.

Now let X be compact Hausdorff with a filter base $\mathfrak{F}$ of dense cozero-sets of X: we write $(X, \mathfrak{F}) \in$ **LSpFi**, the notation and ancestry explained in the Appendix. Each $S \in \mathfrak{F}_\delta$ is dense in X (Baire Category), and is Lindelöf and Čech-complete.

For each $S \in \mathfrak{F}_\delta$, we have these topologies on $C(X)$:

$$\tau_S := \text{the } co \text{ topology on } C(S), \text{ traced on } C(X),$$

$$\sigma_S := \text{the } cz \text{ topology on } C(S), \text{ traced on } C(X).$$

For basic nbds of the group identity = constant function 0 in $C(X)$, we adopt the notation

(*co*) $U(K,\epsilon) = \{g \in C(X) \mid |g| \leqslant \epsilon \text{ on } K\}$ $(K \in \mathcal{K}(S),\ \epsilon \in (0,1))$,
(*cz*) $U(K) = \{g \in C(X) \mid g = 0 \text{ on } K\}$ $(K \in \mathcal{K}(S))$.

Note that S enters here through "$K \in \mathcal{K}(S)$".

We come to the topologies on $C(X)$ considered in this paper. The origins are explained in the Appendix. In the following, $\wedge$ stands for meet (greatest lower bound) in the lattice of topologies on $C(X)$ (the partial order being inclusion). Basics about such $\wedge$'s of topologies are developed in [8].

$$\tau^{\mathfrak{F}} := \bigwedge\{\tau_S \mid S \in \mathfrak{F}_\delta\}; \qquad \sigma^{\mathfrak{F}} := \bigwedge\{\sigma_S \mid S \in \mathfrak{F}_\delta\}.$$

Elementary features of $\wedge$'s imply quickly that each of $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$ has these properties: T_1; inversion $(f \mapsto -f)$, and any translation $(f \mapsto f+g)$, are homeomorphisms; $+$ is separately continuous. (See [2,13].) Consequently, $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$ are homogeneous.

Consider a general meet of topologies on a set, $t = \bigwedge_{i \in I} t_i$. If η_i is a base (at p) for t_i, then $\{\bigcup_{i \in I}\{U_i \mid U_i \in \eta_i\}\}$ is a base (at p) for t. For our $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$, we consider $p = 0$. For example, for $\tau^{\mathfrak{F}}$, a basic nbd of 0 has the form

$$\bigcup\bigl\{U(K_S,\epsilon_S) \bigm| S \in \mathfrak{F}_\delta\bigr\}, \quad \text{where for each } S \in \mathfrak{F}_\delta,\ K_S \in \mathcal{K}(S) \text{ and } \epsilon_S \in (0,1). \tag{$*$}$$

This can be simplified with the following crucial idea:

An adequate family (relative to $\mathfrak{F}$, or $\mathfrak{F}_\delta$) is an $\mathcal{L} \subseteq \mathcal{K}(X)$ for which $[\forall S \in \mathfrak{F}_\delta\ (\mathcal{L} \cap \mathcal{K}(S) \neq \emptyset)]$.

Given adequate $\mathcal{L}$ and $\epsilon \in (0,1)$, we set

$$U(\mathcal{L},\epsilon) := \bigcup\bigl\{U(L,\epsilon) \bigm| L \in \mathcal{L}\bigr\}; \qquad U(\mathcal{L}) := \bigcup\bigl\{U(L) \bigm| L \in \mathcal{L}\bigr\}.$$

Note that, in the first expression, ϵ does not vary.

Proposition 2.1. $\{U(\mathcal{L},\epsilon) \mid \mathcal{L}$ *adequate and* $\epsilon \in (0,1)\}$ *is a base at* 0 *for* $\tau^{\mathfrak{F}}$; $\{U(\mathcal{L}) \mid \mathcal{L}$ *adequate*$\}$ *is a nbd base at* 0 *for* $\sigma^{\mathfrak{F}}$.

Proof. For $\sigma^{\mathfrak{F}}$ there is nothing to prove.

For $\tau^{\mathfrak{F}}$: Each $U(\mathcal{L},\epsilon)$ contains a set of the form $(*)$ above, thus is a nbd. We need to show that each set of the form $(*)$ contains a $U(\mathcal{L},\epsilon)$. Given $U = \bigcup\{U(K_S,\epsilon_S) \mid S \in \mathfrak{F}_\delta\}$ of form $(*)$, for each $n \in \mathbb{N}$, let $\mathcal{L}_U^n = \{L \in \mathcal{K}(X) \mid U(L, \frac{1}{n}) \subseteq U\}$. Clearly, $\mathcal{L}^\circ = \bigcup_{n \in \mathbb{N}} \mathcal{L}_U^n$ is adequate. The point is: there is n_0 such that $\mathcal{L}_U^{n_0}$ is adequate, and clearly, $U(\mathcal{L}_U^{n_0}, \frac{1}{n_0}) \subseteq U$. For if no $\mathcal{L}_U^n$ is adequate, then for every n there is $S_n \in \mathfrak{F}_\delta$ such that $\mathcal{L}_U^n \cap \mathcal{K}(S_n) = \emptyset$. Then $S = \bigcap S_n$ has $\mathcal{L}^\circ \cap \mathcal{K}(S) = \emptyset$, contradicting "$\mathcal{L}^\circ$ is adequate". $\square$

Here are two very simple examples of adequate families for which we have later use.

Examples 2.2. (a) Let $(X, \mathfrak{F}) \in \mathbf{LSpFi}$. For any $p \in \bigcap \mathfrak{F}$, $\{\{p\}\}$ is adequate.

(b) Given Y, thus $(\beta Y, \mathcal{C}) \in \mathbf{LSpFi}$, let $\mathcal{L}^* = \{\{p\} \mid p \in \beta Y \backslash \upsilon Y\}$. This $\mathcal{L}^*$ is adequate iff υY is not both Lindelöf and Čech-complete.

Proof. (a) is obvious.

(b) is because any $S \in \mathcal{C}_\delta$ is Lindelöf and Čech-complete, and $\upsilon Y = \bigcap \mathcal{C}_\delta$ (as discussed before Corollary 1.2). □

We now consider for $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$, first briefly, the Hausdorff property and then, for the rest of the paper, the topological group property. Of course, the latter (and T_1) ⇒ the former (even Tychonoff). $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$ are T_1 just because the meet of T_1-topologies is again T_1, but the meet of Hausdorff topologies need not be [8].

However, to present purposes:

Proposition 2.3.

(a) *Let $(X, \mathcal{F}) \in$ **LSpFi**. If $\bigcap \mathfrak{F}$ is dense in X, then $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$ are Hausdorff.*
(b) *For any $(\beta Y, \mathcal{C})$, $\tau^{\mathcal{C}}$ and $\sigma^{\mathcal{C}}$ are Hausdorff.*

Proof. (b) follows from (a), since $\bigcap \mathcal{C} = \upsilon Y$ is dense in βY.

We prove (a) for $\tau^{\mathfrak{F}}$ ($\sigma^{\mathfrak{F}}$ being even easier).

Since $\tau^{\mathfrak{F}}$ is homogeneous, it is sufficient to separate by open sets 0 from any $f \neq 0$. So given $f \neq 0$, we find $U(\mathcal{L}, \varepsilon)$ as in Proposition 2.1, with disjoint $f + U(\mathcal{L}, \varepsilon)$: Since $\bigcap \mathfrak{F}$ is dense, there is $p \in \bigcap \mathfrak{F}$ where $|f(p)| > 0$. Let $\mathcal{L} = \{\{p\}\}$ and let $\varepsilon = \frac{1}{2}|f(p)|$. This works. □

Proposition 2.3 is a version of 6.8 of [2], there derived as a consequence of a rather opaque criterion that a general $\tau^{\mathfrak{F}}$ be Hausdorff. Also, in 6.5 of [2] appears an example with $\tau^{\mathfrak{F}}$ not Hausdorff and $\bigcap \mathfrak{F} = \emptyset$. The gap between "$\bigcap \mathfrak{F} = \emptyset$" and "$\bigcap \mathfrak{F}$ dense" is largely unexplored, and probably the subject of a future paper similar to the present one. (We note that there is an example in [4] of $\bigcap \mathfrak{F} = \emptyset$ and $\tau^{\mathfrak{F}}$ Hausdorff.)

Now we turn to the topological group property of $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$. The criterion is just a translation into adequate families of the basic principle [14]: For a group $(G, +)$ with topology t, t is a group topology iff ∀ (basic) nbd U of 0 ∃ (basic) nbd V of 0 ($V + V \subseteq U$).

The rest of this section refers to a fixed $(X, \mathfrak{F}) \in \mathbf{LSpFi}$.

Definition 2.4. Let $\mathcal{L}$, $\mathcal{M}$ be adequate families. $\mathcal{L} \overset{o}{\prec} \mathcal{M}$ (respectively $\mathcal{L} \overset{z}{\prec} \mathcal{M}$) means: $\forall M_1, M_2 \in \mathcal{M}$ ∀ open (respectively zero-) sets $U_i \supseteq M_i$, there is an $L \in \mathcal{L}$ with $L \subseteq U_1 \cap U_2$.

Theorem 2.5. $\tau^{\mathfrak{F}}$ *(respectively, $\sigma^{\mathfrak{F}}$) is a group topology iff for every adequate $\mathcal{L}$ there is an adequate $\mathcal{M}$ with $\mathcal{L} \overset{o}{\prec} \mathcal{M}$ (respectively, $\mathcal{L} \overset{z}{\prec} \mathcal{M}$).*

Proof. ($\tau^{\mathfrak{F}}$). Suppose $\tau^{\mathfrak{F}}$ is a group topology. Let $\mathcal{L}$ be adequate and $\epsilon \in (0,1)$. There are adequate $\mathcal{M}$ and $\delta \in (0,1)$ with $U(\mathcal{M}, \delta) + U(\mathcal{M}, \delta) \subseteq U(\mathcal{L}, \epsilon)$. This implies $\mathcal{L} \overset{o}{\prec} \mathcal{M}$: If not, there are $\mathcal{M} \ni M_i \subseteq$ open U_i with $L \nsubseteq U_1 \cap U_2$ $\forall L \in \mathcal{L}$. Thus $\forall L$ $\exists x_L \in L \cap (X \backslash (U_1 \cap U_2)) = L \cap ((X \backslash U_1) \cup (X \backslash U_2))$. Let $f_i \in C(X)$ have $f_i(M_i) = 0$ and $f_i(X \backslash U_i) = 2$. Then, $|f_i| = 0 < \delta$ on M_i, so $f_i \in U(\mathcal{M}, \delta)$, while $(f_1 + f_2)(x_L) \geqslant 2 > \epsilon$ for every L, so $f_1 + f_2 \notin U(\mathcal{L}, \epsilon)$.

Conversely: Suppose for every adequate $\mathcal{L}$ there is an adequate $\mathcal{M}$ with $\mathcal{L} \overset{o}{\prec} \mathcal{M}$. Take a basic nbd of 0, $U(\mathcal{L}, \epsilon)$. Take $\mathcal{M}$ with $\mathcal{L} \overset{o}{\prec} \mathcal{M}$, and let $\delta = \frac{\epsilon}{4}$. Let $f_1, f_2 \in U(\mathcal{M}, \delta)$; $\exists M_i \in \mathcal{M}$ ($|f_i| \leqslant \delta$ on M_i). Let $U_i = \{x \mid |f_i(x)| < 2\delta\}$, so $\exists L \in \mathcal{L}$ with $L \subseteq U_1 \cap U_2$. Then, $|f_1 + f_2| \leqslant \epsilon$ on L, so $f_1 + f_2 \in U(\mathcal{L}, \epsilon)$.

($\sigma^{\mathfrak{F}}$). Suppose $\sigma^{\mathfrak{F}}$ is a group topology. Let $\mathcal{L}$ be adequate. There is adequate $\mathcal{M}$ with $U(\mathcal{M}) + U(\mathcal{M}) \subseteq U(\mathcal{L})$. This implies $\mathcal{L} \overset{z}{\prec} \mathcal{M}$: If not, there are $\mathcal{M} \ni M_i \subseteq$ zero-set U_i with $L \nsubseteq U_1 \cap U_2$ $\forall L \in \mathcal{L}$. Thus $\forall L$ $\exists x_L \in L \cap (X \backslash (U_1 \cap U_2))$. Let $f_i \in C(X)^+$ have $Zf_i = U_i$. Then $|f_i| = 0$ on M_i, so $f_i \in U(\mathcal{M})$, while $(f_1 + f_2)(x_L) > 0$ for every L, so $f_1 + f_2 \notin U(\mathcal{L})$.

Conversely: Suppose for every adequate $\mathcal{L}$ there is an adequate $\mathcal{M}$ with $\mathcal{L} \overset{z}{\prec} \mathcal{M}$. Take a basic nbd of 0, $U(\mathcal{L})$. Take $\mathcal{M}$ with $\mathcal{L} \overset{z}{\prec} \mathcal{M}$. Let $f_1, f_2 \in U(\mathcal{M})$; $\exists M_i \in \mathcal{M}$ ($|f_i| = 0$ on M_i). Let $U_i = Zf_i$, so $\exists L \in \mathcal{L}$ with $L \subseteq U_1 \cap U_2$. Then, $|f_1 + f_2| = 0$ on L, so $f_1 + f_2 \in U(\mathcal{L})$. □

In Definition 2.4 and Theorem 2.5, the open sets and the zero-sets U_i are difficult to handle, especially for our favorites $(\beta Y, \mathcal{C})$, where a certain amount of floating around in $\beta Y \backslash Y$ is taking place. Our successes involve a third simpler condition $\mathcal{L} \prec \mathcal{M}$, then in Sections 3, 4, pushing the issues in βY down into Y.

Definition 2.6. Let $\mathcal{L}$, $\mathcal{M}$ be adequate families. $\mathcal{L} \prec \mathcal{M}$ means: $\forall M_1, M_2 \in \mathcal{M}$, there is an $L \in \mathcal{L}$ with $L \subseteq M_1 \cap M_2$.

Proposition 2.7.

(a) $\mathcal{L} \prec \mathcal{M} \Rightarrow \mathcal{L} \overset{z}{\prec} \mathcal{M} \Rightarrow \mathcal{L} \overset{o}{\prec} \mathcal{M}$.
(b) *Each of the following implies the next*:
$\forall$ adequate $\mathcal{L}$ $\exists$ adequate $\mathcal{M}$ ($\mathcal{L} \prec \mathcal{M}$); $\sigma^{\mathfrak{F}}$ is a group topology; $\tau^{\mathfrak{F}}$ is a group topology.

Proof. (b) follows from (a) and Theorem 2.5.

(a) The first implication is obvious. For the second, recall that, if compact $M \subseteq$ open U in Tychonoff X, then some $f \in C(X)$ has $M \subseteq Z(f)$ and $X \backslash U \subseteq coz\, f$. Now suppose $\mathcal{L} \overset{z}{\prec} \mathcal{M}$. Take $\mathcal{M} \ni M_i \subseteq$ open U_i, choose $f_i \in C(X)$ with $M_i \subseteq Z(f_i) \subseteq U_i$, and then choose $\mathcal{L} \ni L \subseteq Z(f_1) \cap Z(f_2) \subseteq U_1 \cap U_2$. □

Let $\overset{*}{\prec}$ stand for any of $\prec$, $\overset{z}{\prec}$, $\overset{o}{\prec}$. In considering issues of $\mathcal{L} \overset{*}{\prec} \mathcal{M}$, it sometimes happens (e.g., Section 3) that the situation becomes clearer if we convert to adequate maps as follows.

An *adequate map* is a function $\mathcal{L} : \mathfrak{F}_\delta \to \mathcal{K}(X)$ for which $\forall S$ $(\mathcal{L}(S) \in \mathcal{K}(S))$; so the range $\mathcal{L}(\mathfrak{F}_\delta)$ is an adequate family. For an adequate family $\mathcal{L}$, we create an adequate map by choice: $\forall S \in \mathfrak{F}_\delta$ $(\mathcal{L} \cap \mathcal{K}(S) \neq \emptyset)$, so choose $\mathcal{L}(S) \in \mathcal{L} \cap \mathcal{K}(S)$.

For adequate maps, $\mathcal{L} \overset{*}{\prec} \mathcal{M}$ means, for the ranges $\mathcal{L}(\mathfrak{F}_\delta) \overset{*}{\prec} \mathcal{M}(\mathfrak{F}_\delta)$.

Then, Theorem 2.5 and Proposition 2.7 convert to exactly the same statements for adequate maps, which we skip writing down.

3. $C(\beta\lambda D(\omega_1))$ is a topological group

By the title of this section, we mean Theorem 1.3, which we shall prove. Abbreviate $D(\omega_1)$ to D. Recall $\lambda D = D \cup \lambda$ defined in Section 1. This λD is Lindelöf, a P-space (G_δ-sets are open), thus $\beta\lambda D$ is basically disconnected and thus zero-dimensional. (Refer to [11] if needed.)

To deal with the condition $\mathcal{L} \overset{*}{\prec} \mathcal{M}$ ($\overset{*}{\prec}$ is any of $\prec$, $\overset{z}{\prec}$, $\overset{o}{\prec}$) for $(\beta\lambda D, \mathcal{C})$, we shall (1) make a general reduction to "basic adequate families"; (2) apply (1) to $(\beta Y, \mathcal{C})$ for Y any Lindelöf P-space; (3) further specialize (2) to $Y = \lambda D$; (4) convert the issues "$\mathcal{L} \overset{*}{\prec} \mathcal{M}$?" to a set-theoretic problem about D; (5) solve that problem, thus proving Theorem 1.3.

For any set E, and family $\mathcal{A}$ of subsets of E: $\mathcal{B}$ is *coinitial* (respectively, *cofinal*) in $\mathcal{A}$ means: $\mathcal{B} \subseteq \mathcal{A}$, and $\forall A \in \mathcal{A}$ $\exists B \in \mathcal{B}$ with $B \subseteq A$ (respectively, $B \supseteq A$).

We shall create the "basic adequate families" referred to above by combining a coinitial subfamily $\mathfrak{S}$ of $\mathfrak{F}_\delta$ (every $\mathfrak{F}_\delta$ contains an $S \in \mathfrak{S}$) with, for each $S \in \mathfrak{S}$, a cofinal subfamily $\mathcal{K}_0(S)$ of $\mathcal{K}(S)$ (every compact set in S is contained in a $K \in \mathcal{K}_0(S)$).

Lemma 3.1. *Let $(X, \mathfrak{F}) \in$* **LSpFi**. *Suppose $\mathfrak{S}$ is coinitial in $\mathfrak{F}_\delta$, and $\forall S \in \mathfrak{S}$, we have $\mathcal{K}_0(S)$ cofinal in $\mathcal{K}(S)$. Then*

(a) $\mathbb{L} = \mathbb{L}(\mathfrak{S}, \mathcal{K}_0) \equiv \{\mathcal{L} \subseteq \mathcal{K}(X) \mid \forall S \in \mathfrak{S},\ \mathcal{L} \cap \mathcal{K}_0(S) \neq \emptyset\}$ *consists of adequate families.*
(b) $\{U(\mathcal{L}, \epsilon) \mid \mathcal{L} \in \mathbb{L},\ \epsilon \in (0,1)\}$ *is a basis at* 0 *for* $\tau^{\mathfrak{F}}$, *and* $\{U(\mathcal{L}) \mid \mathcal{L} \in \mathbb{L}\}$ *is a basis at* 0 *for* $\sigma^{\mathfrak{F}}$.
(c) $[\forall \mathcal{L}\ \exists \mathcal{M}\ (\mathcal{L} \overset{*}{\prec} \mathcal{M})]$ *holds in the collection of all adequate families iff it holds in* $\mathbb{L}$ *(for each of the three* $\overset{*}{\prec}$*).*
(d) *Suppose further that $\forall S \in \mathfrak{S}$, $\mathcal{K}_0(S) \subseteq clop\, X$. Then the three conditions $[\forall \mathcal{L} \in \mathbb{L}\ \exists \mathcal{M} \in \mathbb{L}\ (\mathcal{L} \overset{*}{\prec} \mathcal{M})]$ are equivalent, and: $\sigma^{\mathfrak{F}}$ is a group topology iff $\tau^{\mathfrak{F}}$ is iff $[\forall \mathcal{L}\ \exists \mathcal{M}\ (\mathcal{L} \prec \mathcal{M})]$ in $\mathbb{L}$.*

There are no surprises in the proof of Lemma 3.1. One just works through the details. We omit this.

Lemma 3.2. *Consider $(\beta Y, \mathcal{C})$, for Y a Lindelöf P-space.*

(a) $\mathfrak{S} = \mathcal{C}$ *is coinitial in* $\mathcal{C}_\delta$.
(b) $\forall S \in \mathcal{C}\ \exists \mathcal{K}_0(S) \subseteq clop\, \beta Y$ *which is cofinal in* $\mathcal{K}(S)$.

Thus, Lemma 3.1 *applies to* $\mathbb{L} = \mathbb{L}(\mathfrak{S}, \mathcal{K}_0)$.

Proof. (The closures indicated in the following are in βY.) (a) Take $\{F_n\}_{\mathbb{N}} \subseteq \mathcal{C}$, so $F = \bigcap F_n \in \mathcal{C}_\delta$. For every $y \in Y$ and $n \in \mathbb{N}$, there is $U_n \in clop\, \beta Y$ with $y \in U_n \subseteq F_n$ (since F_n is open and βY is zero-dimensional). Then $U_y = \bigcap U_n$ is a closed G_δ in βY, so $U_y \cap Y \in clop\, Y$ (since Y is a P-space), and $U_y \cap Y \subseteq U_n \subseteq F_n$ $\forall n$. Thus $V_y \equiv \overline{U_y \cap Y} \subseteq U_n \subseteq F_n$ $\forall n$, so $V_y \subseteq F$, and $V_y \in clop\, \beta Y$ (since for any Y, $G \in clop\, Y \Rightarrow \overline{G} \in clop\, \beta Y$). $\{V_y \mid y \in Y\}$ covers Lindelöf Y, so there is a countable subcover $\{V_{y(n)}\}_{\mathbb{N}}$. Then, $S \equiv \bigcup_{\mathbb{N}} V_{y(n)} \in \mathcal{C}$ and $S \subseteq F$.

(b) Let $S \in \mathcal{C}$, say $S = coz\, f$ for $f \in C(\beta Y)$. Then, $\mathcal{K}_0(S) \equiv \{\overline{f^{-1}(\frac{1}{n}, +\infty)} \mid n \in \mathbb{N}\}$ is cofinal in $\mathcal{K}(S)$ (since, if $K \in \mathcal{K}(S)$, $f|K$ is bounded away from 0 because K is compact). And, $\mathcal{K}_0(S) \subseteq clop\, \beta Y$, since βY is basically disconnected. □

We note here that we do not know if $\tau^{\mathcal{C}}$ and $\sigma^{\mathcal{C}}$ are group topologies for $(\beta Y, \mathcal{C})$, Y any Lindelöf P-space. (I.e., is Theorem 1.3 true replacing $\lambda D(\omega_1)$ by any such Y?) Lemma 3.2(b) is included toward that issue; only the more explicit version Lemma 3.3(b) is used below.

In the following, $D^{<\omega} = \{F \subseteq D \mid |F| < \omega\}$.

Lemma 3.3. *Now consider $(\beta\lambda D, \mathcal{C})$. Let $coc\, D \equiv \{B \subseteq D \mid |D \backslash B| \leqslant \omega\}$.*

(a) *$\mathfrak{S} = \{\overline{B} \cup D \mid B \in coc\, D\}$ is coinitial in $\mathcal{C}$, thus too in $\mathcal{C}_\delta$.*
(b) *For $\overline{B} \cup D = S \in \mathfrak{S}$, let $\mathcal{K}_0(S) \equiv \{\overline{B} \cup F \mid F \in D^{<\omega}\}$. Then, $\mathcal{K}_0(S) \subseteq clop\, \beta\lambda D$, and $\mathcal{K}_0(S)$ is cofinal in $\mathcal{K}(S)$.*

Thus, Lemma 3.1 *applies to $\mathbb{L} = \mathbb{L}(\mathfrak{S}, \mathcal{K}_0)$.*

Proof. (a) $\mathcal{C}$ is coinitial in $\mathcal{C}_\delta$, by Lemma 3.2. We show $\mathfrak{S}$ is coinitial in $\mathcal{C}$. First, each $\overline{B} \cup D \in \mathcal{C}$: $D \backslash B = \{y_1, y_2, \ldots\}$. Define $g \in C(\beta\lambda D)$ as: $g(y_n) = \frac{1}{n}$; $g|B = 1$; extend g over $\beta\lambda D$, achieving $coz\, g = \overline{B} \cup D$.

Let $F \in \mathcal{C}$, say $F = coz\, f$ for $f \in C(\beta\lambda D)$. Since λ is a P-point of $\beta\lambda D$ (see [11]), $U \equiv f^{-1}(\{f(\lambda)\}) \in clop\, \beta\lambda D$, so $B \equiv U \cap \lambda D$ is a nbd of λ in λD, so $|D \backslash B| \leqslant \omega$. Finally, $\overline{B} \cup D \subseteq U \cup D \subseteq coz\, f$.

(b) Let $S = \overline{B} \cup D$. $K \cup \{\lambda\} \in clop\, \lambda D$, so $\overline{B} \in clop\, \beta\lambda D$. Since points of D are isolated in $\beta\lambda D$, $D^{<\omega} \subseteq clop\, \beta\lambda D$. So $\mathcal{K}_0(S) \subseteq clop\, \beta\lambda D$.

Let $K \in \mathcal{K}(S)$. Then $K \cap D = (K \cap (D \backslash B)) \cup (K \cap B)$, so $K = K \cap S = (K \cap \overline{B}) \cup (K \cap D) = (K \cap \overline{B}) \cup (K \cap (D \backslash B)) \cup (K \cap B) = (K \cap \overline{B}) \cup (K \cap (D \backslash B)) \subseteq \overline{B} \cup F$ for $F = K \cap (D \backslash B)$. For g as in (a), g is bounded away from 0 on K, thus on F, so F is finite. □

Proof of Theorem 1.3. Take $\mathbb{L} = \mathbb{L}(\mathfrak{S}, \mathcal{K}_0)$ from Lemma 3.3. Convert to the associated family of adequate maps, as described in the end of Section 2, still denoted $\mathbb{L}$: the maps $\mathcal{L} \in \mathbb{L}$ are functions $\mathcal{L} : \mathfrak{S} \to \mathcal{K}(\beta\lambda D)$ with $\mathcal{L}(S) \in \mathcal{K}_0(S)$ $\forall S \in \mathfrak{S}$.

By Lemma 3.1, we can be concerned only with $\prec$, and we are to show

$$\left[\forall \mathcal{L} \in \mathbb{L}\ \exists \mathcal{M} \in \mathbb{L}\ (\mathcal{L} \prec \mathcal{M})\right].$$

Now $\mathcal{L} \prec \mathcal{M}$ means $\mathcal{L}(\mathfrak{S}) \prec \mathcal{M}(\mathfrak{S})$ for the ranges, which means:

$$\forall S_1, S_2\ \exists S_3\ \left(\mathcal{L}(S_3) \subseteq \mathcal{M}(S_1) \cap \mathcal{M}(S_2)\right).$$

Keep in mind that $S \in \mathfrak{S}$ means $S = \overline{B} \cup D$ for $B \in coc\, D$, and for $\mathcal{L} \in \mathbb{L}$, $\mathcal{L}(S) \in \mathcal{K}_0(S)$, so either

(i) $\mathcal{L}(S) = \overline{B} \cup F$ for $F \in D^{<\omega}$, or
(ii) $\mathcal{L}(S) = F$ for $F \in D^{<\omega}$.

Now let $\mathcal{L} \in \mathbb{L}$.

If (ii) ever happens, i.e., $\exists S_0$ with $\mathcal{L}(S_0) = F_0 \in D^{<\omega}$, just define $\mathcal{M}(S) = F_0$ $\forall S$. Then $\mathcal{L} \prec \mathcal{M}$.

Otherwise, (i) always happens. Consider $\mathbb{L}^{(i)} \equiv \{\mathcal{L} \in \mathbb{L} \mid \text{(i) always happens}\}$. $\mathcal{L} \in \mathbb{L}^{(i)}$ means: $\forall S = \overline{B} \cup D$, $\mathcal{L}(S) = \overline{B} \cup F$ for some $F \in D^{<\omega}$. We are to show $[\forall \mathcal{L} \in \mathfrak{L}^{(i)}\ \exists \mathcal{M} \in \mathfrak{L}^{(i)}\ (\mathcal{L} \prec \mathcal{M})]$.

Now convert from $\mathfrak{L}^{(i)}$ to the family *FEM* of "finitely enlarging maps" in $coc\, D$: $\mathcal{P} \in FEM$ means $\mathcal{P} : coc\, D \to coc\, D$, $\forall B$ $(\mathcal{P}(B) \supseteq B)$ and $|\mathcal{P}(B) \backslash B| < \omega$. The conversion is the function $p : \mathfrak{L}^{(i)} \to FEM$ defined as: For $\mathcal{L} \in \mathbb{L}^{(i)}$, so $\mathcal{L}(\overline{B} \cup D) = \overline{B} \cup F$, $p\mathcal{L}(B) \equiv B \cup F$. It is easily seen that p is one-to-one and onto *FEM*.

In *FEM*, define $\mathcal{P} \prec \mathcal{R}$ as: $\forall B_1, B_2 \in coc\, D$, $\exists B_3 \in coc\, D$ $(\mathcal{P}(B_3) \subseteq \mathcal{R}(B_1) \cap \mathcal{R}(B_2))$. □

Lemma 3.4. *For $B_1, B_2, B_3 \in coc\, D$, and $F_1, F_2, F_3 \in D^{<\omega}$,* (a) *$B_3 \cup F_3 \subseteq (B_1 \cup F_1) \cap (B_2 \cup F_2)$ iff* (b) *$\overline{B}_3 \cup F_3 \subseteq (\overline{B}_1 \cup F_1) \cap (\overline{B}_2 \cup F_2)$ (where $\overline{(\ )}$ is closure in $\beta\lambda D$).*

Proof. Suppose (a). Then $B_3 \cup F_3 \subseteq (\overline{B}_1 \cup F_1) \cap (\overline{B}_2 \cup F_2)$, the right side is closed, and thus contains $\overline{B}_3 \cup F_3$.

Conversely: For a generic $B \cup F$, we have $(\overline{B} \cup F) \cap D = B \cup F$. Now assume (b), and intersect this inclusion with D to get (a). □

Corollary 3.5.

(a) *In $\mathbb{L}^{(i)}$ $[\mathcal{L} \prec \mathcal{M}]$ iff in FEM $[p\mathcal{L} \prec p\mathcal{M}]$.*
(b) *In $\mathbb{L}^{(i)}$ $[\forall \mathcal{L}\ \exists \mathcal{M}\ (\mathcal{L} \prec \mathcal{M})]$ iff in FEM $[\forall \mathcal{P}\ \exists \mathcal{R}\ (\mathcal{P} \prec \mathcal{R})]$.*

Thus the following proves Theorem 1.3.

Theorem 3.6. *In FEM $[\forall \mathcal{P}\ \exists \mathcal{R}\ (\mathcal{P} \prec \mathcal{R})]$.*

Proof. Identify D with the ordinals $\{\alpha \mid \alpha < \omega_1\}$. A *club* is a closed unbounded set in D.

Fix $\mathcal{P} \in FEM$. For $\alpha < \omega_1$, set $P_\alpha \equiv \mathcal{P}(\omega_1 - \alpha) \cap \alpha \in [\alpha]^{<\omega}$. Write $[\omega_1]^{<\omega} = \{F_\beta \mid \beta < \omega_1\}$.

(a) There is a club $C \subseteq \omega_1$ for which: $\alpha \in C \Rightarrow [\alpha]^{<\omega} \subseteq \{F_\beta \mid \beta < \alpha\}$.

The proof is below. Knowing (a) we proceed.

For $\alpha \in C$, $P_\alpha \in [\alpha]^{<\omega}$, so $\exists \beta < \alpha$ such that $P_\alpha = F_\beta$. Choose such β and call it $\varphi(\alpha)$. This defines $\varphi : C \to \omega_1$ with $\varphi(\alpha) < \alpha$ $\forall \alpha \in C$.

(b) $\exists \beta_0 < \omega_1$ such that $\varphi^{-1}(\beta_0)$ is unbounded.

(Indeed, $\varphi^{-1}(\beta_0)$ is "stationary" ($\equiv$ meets every club). This is the pressing-down lemma ($=$ Fodor's Theorem/Lemma). See [9, p. 162].)

Let $E = \varphi^{-1}(\beta_0)$: $\alpha \in E$ means $\varphi(\alpha) = \beta_0$, which means $P_\alpha = F_{\beta_0}$.

Define $\mathcal{R} \in FEM$ as: $\mathcal{R}(B) = B \cup F_{\beta_0}$. Showing $\mathcal{P} \prec \mathcal{R}$: For $B_1, B_2 \in coc\, D$, $\mathcal{R}(B_1) \cap \mathcal{R}(B_2) = (B_1 \cap B_2) \cup F_{\beta_0}$. Now, $\exists \alpha_i < \omega_1$ s.t. $B_i \subseteq \omega_1 - \alpha_i$, so $\exists \alpha \in E$ with $\alpha \geqslant \alpha_1 \vee \alpha_2$, and $\omega_1 - \alpha \subseteq B_1 \cap B_2$, and $P_\alpha = F_{\beta_0}$. Thus,

$$\mathcal{P}(\omega_1 - \alpha) = (\omega_1 - \alpha) \cup F_{\beta_0} \subseteq (B_1 \cap B_2) \cup F_{\beta_0} = \mathcal{R}(B_1) \cap \mathcal{R}(B_2).$$

We prove (a): Define $\{\alpha_\gamma \mid \gamma < \omega_1\}$ as follows: (1) $\alpha_0 = 0$; (2) with α_δ defined, $\alpha_{\delta+1} = \mu(\alpha_\delta)$, where $\mu : \omega_1 \to \omega_1$ and $\forall \alpha < \omega_1$ $\mu(\alpha)$ is the least ordinal $> \alpha$, such that $[\alpha]^{<\omega} \subseteq \{F_\beta \mid \beta < \mu(\alpha)\}$; (3) for γ limit ordinal, and α_δ defined $\forall \delta < \gamma$, we define $\alpha_\gamma = \bigvee_{\delta<\gamma} \alpha_\delta$. Now induct over ω_1.

It is clear that $\gamma < \gamma'$ iff $\alpha_\gamma < \alpha_{\gamma'}$. So $\{\alpha_\gamma \mid \gamma < \omega_1\}$ is unbounded, and is closed: Given $\{\alpha_{\gamma_n}\}_{\mathbb{N}}$, $\bigvee_{\mathbb{N}} \alpha_{\gamma_n} = \alpha_\gamma$ for $\gamma = \bigvee_{\mathbb{N}} \gamma_n$ by definition. Note that [if $\alpha = \bigvee_I \alpha_i$ then $[\alpha]^{<\omega} = \bigcup_I [\alpha_i]^{<\omega}$]. Thus γ limit $\Rightarrow [\alpha_\gamma]^{<\omega} \subseteq \{F_\beta \mid \beta < \alpha_\gamma\}$. (Since $F \in [\alpha_\gamma]^{<\omega} \Rightarrow \exists \delta < \gamma$ with $F \in [\alpha_\gamma]^{<\omega} \subseteq \{F_\beta \mid \beta < \alpha_{\delta+1}\} \subseteq \{F_\beta \mid \beta < \alpha_\gamma\}$.)

Then let $C = \{\gamma \mid \gamma \text{ limit} < \omega_1\}$. This too is a club. □

4. $C(\beta D(\omega_1))$ is not a topological group

By the title of this section, we mean the first sentence of Theorem 1.4. As before, let $D = D(\omega_1)$. By Proposition 2.7, it is enough to show

Theorem 4.1. *On $C(\beta D)$, $\tau^{\mathcal{C}}$ fails Theorem* 2.5, *and is not a group topology.*

After proving this, we sketch the extension to certain $Y \supseteq D$ as mentioned in Theorem 1.4.

We proceed with the proof of Theorem 1.4 considering a general $(\beta Y, \mathcal{C})$, restricting generality at each step as forced, finally down to $Y = D$.

Given Y, let $\mathcal{L}^* = \{\{p\} \mid p \in \beta Y \backslash \upsilon Y\}$, as in Examples 2.2. For $Y = D = \upsilon Y$, $\mathcal{L}^*$ is adequate, and we shall ultimately show [$\nexists$ adequate $\mathcal{M}$ ($\mathcal{L}^* \overset{o}{\prec} \mathcal{M}$)]. Thus Theorem 4.1 follows.

Lemma 4.2. *Given Y, thus $(\beta Y, \mathcal{C}) \in$* **LSpFi**:

(a) *(See Examples 2.2.) $\mathcal{L}^*$ is adequate iff υY is not both Lindelöf and Čech-complete.*
(b) *Suppose $\mathcal{L}^*$ is adequate, and $(C(\beta Y), \tau^{\mathcal{C}})$ is a topological group. Then ($\exists$ adequate $\mathcal{M}$ ($\mathcal{L}^* \overset{o}{\prec} \mathcal{M}$)).*
(c) *Suppose $\mathcal{L}^*$ is adequate, and let $\mathcal{M}$ be adequate. [$\mathcal{L}^* \overset{o}{\prec} \mathcal{M}$] fails iff $\exists M_1, M_2 \in \mathcal{M}$ with nbds U_i of M_i with ($U_1 \cap U_2 \subseteq \upsilon Y$) iff $\exists S_1, S_2 \in \mathcal{C}_\delta$, and $M_i \in \mathcal{M} \cap \mathcal{K}(S_i)$ with nbds U_i of M_i with ($U_1 \cap U_2 \subseteq \upsilon Y$).*

Proof. (b) follows from Theorem 2.5.

(c) [$\mathcal{L}^* \overset{o}{\prec} \mathcal{M}$] fails means $\exists M_i$ and U_i with $p \in \beta Y \backslash \upsilon Y \Rightarrow p \nsubseteq U_1 \cap U_2$, i.e., $(\beta Y \backslash \upsilon Y) \cap (U_1 \cap U_2) = \emptyset$, i.e., $U_1 \cap U_2 \subseteq \upsilon Y$. The third condition in (c) is just a translation. □

Paraphrasing Lemma 4.2(c), [$\mathcal{L}^* \overset{o}{\prec} \mathcal{M}$] fails iff $\exists S_1, S_2 \in \mathcal{C}_\delta$ witnessing that, and $\forall \mathcal{M}$ [$\mathcal{L}^* \overset{o}{\prec} \mathcal{M}$] fails iff $\exists \mathfrak{S} \subseteq \mathcal{C}_\delta$, pairs from which witness each of the failures. Call such $\mathfrak{S}$ a "witness".

Lemma 4.3. *Given Y: Suppose $\mathcal{L}^*$ is adequate, and $\mathfrak{S} \subseteq \mathcal{C}_\delta$ has the properties*:

(a) *each $S \in \mathfrak{S}$ has a "hemicompact representation" $S = \bigcup_{n<\omega} S^n$ (meaning each $S^n \in \mathcal{K}(S)$, and $K \in \mathcal{K}(S) \Rightarrow K \subseteq S^n$ for some n), and*
(b) *$\forall f \in \omega^{\mathfrak{S}}$ $\exists S_1, S_2 \in \mathfrak{S}$ and nbds U_i of $S_i^{f(S_i)}$ with $U_1 \cap U_2 \subseteq Y$.*

Then, $\mathfrak{S}$ is a witness.

Proof. Let $\mathcal{M}$ be adequate. If $S \in \mathfrak{S}$, there is $M(S) \in \mathcal{M} \cap \mathcal{K}(S)$, and so there is a first integer $f(S)$ with $M(S) \subseteq S^{f(S)}$. This defines $f \in \omega^{\mathfrak{S}}$. Then, (b) says "$\exists S_1, S_2 \in \mathfrak{S} \ldots$", showing [$\mathcal{L}^* \overset{o}{\prec} \mathcal{M}$] fails by Lemma 4.2(c). □

Note. The notation $S = \bigcup S^n$ in Lemma 4.3 refers to a distinguished countable decomposition of S. Versions of this notation appear below in Lemma 4.4, Corollary 4.5, and Theorem 4.6.

Lemma 4.4. *Suppose Y is locally compact and realcompact and $\mathcal{L}^*$ is adequate. (Since Y is Čech-complete, this means Y is not Lindelöf.)*

(a) *Let $M_1, M_2 \in \mathcal{K}(\beta Y)$. $\exists$ nbds U_i $(U_1 \cap U_2 \subseteq Y)$ iff $M_1 \cap M_2 \subseteq Y$.*
(b) *Let $\mathcal{M}$ be adequate. $[\mathcal{L}^* \overset{o}{\prec} \mathcal{M}]$ fails iff $\exists S_1, S_2 \in \mathfrak{S}$ and $M_i \in \mathcal{K}(S_i)$ with $M_1 \cap M_2 \subseteq Y$.*
(c) *Let $\mathfrak{S} \subseteq \mathcal{C}_\delta$. Suppose Lemma 4.3(a), and: $\forall f \in \omega^{\mathfrak{S}}$ $\exists S_1, S_2 \in \mathfrak{S}$ $(S_1^{f(S_1)} \cap S_2^{f(S_2)} \subseteq Y)$. Then, $\mathfrak{S}$ is a witness.*

Proof. (b) follows from (a), and (c) follows from (a), (b), and Lemma 4.3.

Showing (a): $\Rightarrow$ is obvious. $\Leftarrow$: $M_1 \cap M_2 \subseteq Y$ and Y locally compact $\Rightarrow$ $M_1 \cap M_2$ and $M_i \cap (\beta Y \backslash \upsilon Y)$ are three disjoint compact sets. So there are disjoint nbds V_0, V_i and we can take $V_0 \subseteq Y$. Then $U_i = V_0 \cup V_i$ are nbds of M_i. $\square$

Now consider uncountable discrete Y of nonmeasurable cardinal, so Y is realcompact [11]. So: $\mathcal{L}^*$ is adequate and Y is locally compact. βY is zero-dimensional and each clopen set is of the form $\overline{A}$ for $A \subseteq Y$. Each $S \in \mathcal{C}$ is locally compact and σ-compact and thus has a hemicompact representation $S = \bigcup_{n<\omega} \overline{A^n}$ with $\bigcup_{n<\omega} A^n = Y$. Whenever $\bigcup_{n<\omega} A^n = Y$, $\bigcup_{n<\omega} \overline{A^n} \in \mathcal{C}$.

Let $\mathfrak{J}$ be an index set, and let $\mu = \{A_\alpha^n \mid \alpha \in \mathfrak{J},\ n < \omega\}$ be a family of subsets of Y with the property: $\forall \alpha$ $(\bigcup_{n<\omega} A_\alpha^n = Y)$. We call such μ a "matrix". For $\alpha \in \mathfrak{J}$, let $S_\alpha = \bigcup_{n<\omega} \overline{A_\alpha^n}$, and let $\mathfrak{S}(\mu) = \{S_\alpha \mid \alpha \in \mathfrak{J}\}$. So $\mathfrak{S}(\mu) \subseteq \mathcal{C}$. If $\mathfrak{S}(\mu)$ is a witness, we call μ a witness.

Consider Lemma 4.4(c) for such $\mathfrak{S}(\mu)$. The parenthetical condition takes the form $\overline{A_\alpha^{f(\alpha)}} \cap \overline{A_\beta^{f(\beta)}} \subseteq Y$ for $A_\alpha^{f(\alpha)}, A_\beta^{f(\beta)} \in \mu$. Now, Y is discrete, and for $A, B \subseteq Y$, $\overline{A} \cap \overline{B} \subseteq Y$ iff $A \cap B$ is finite, and Lemma 4.4(c) becomes:

Corollary 4.5. *Suppose Y is uncountable discrete. Suppose $\mu = (A_\alpha^n)$ is a matrix as above for which: $\forall f \in \omega^{\mathfrak{J}}$ $\exists \alpha, \beta$ $(|A_\alpha^{f(\alpha)} \cap A_\beta^{f(\beta)}| < \omega)$. Then, μ is a witness, and on $C(\beta Y)$, $\tau^{\mathcal{C}}$ is not a group topology.*

Corollary 4.5 and the following prove Theorem 4.1.

Theorem 4.6. *For $Y = D$ there is a matrix μ satisfying Corollary 4.5.*

Proof. We use an Aronszajn tree (A-tree). We refer to [16], for some basic facts, then make a construction.

(a) A tree is a poset $(T, <)$ such that $\forall x$ $\downarrow x \equiv \{y \mid y < x\}$ is well ordered (by $<$). The α-level $T_\alpha = \{x \mid \downarrow x$ has order type $\alpha\}$. The height of $T = \min\{\alpha \mid T_\alpha = \emptyset\}$. So $T = \dot{\bigcup}\{T_\alpha \mid \alpha < \text{height}\}$. A branch is a maximal chain (in T). An A-tree is a tree $(T, <)$ of height ω_1, all $|T_\alpha| \leqslant \omega$, no uncountable branches. So $|T| = \omega_1$. We identify T with the countable ordinals, i.e., $T = \omega_1$. For $\gamma \in T$, its level $\lambda(\gamma) =$ the unique α $(\gamma \in T_\alpha)$. For $\lambda(\gamma) > \alpha$ $\exists$ unique $x \in \downarrow\gamma \cap T_\alpha$; that x is denoted $p_\alpha(\gamma)$ (α-level predecessor of γ; or, projection of γ to α-level).

(Do not confuse the tree order $<$ with ordinal order $<$.)

(b) [16, 22.3] There is an A-tree.

(c) [16, 24.2] Let $(T, <)$ be an A-tree. If $\mathcal{W}$ is an uncountable family of pairwise disjoint finite subsets of T, then $\exists S, S' \in \mathcal{W}$ $(x \in S,\ x' \in S' \Rightarrow x$ and x' are incomparable).

(d) (The theorem) On T $(= D)$, there is μ satisfying Corollary 4.5.

Take an A-tree $(\omega_1, <)$. $\forall \alpha$: $|T_\alpha| \leqslant \omega$ so there is a one-to-one map $T_\alpha \xrightarrow{\psi_\alpha} \omega$ $(= \mathbb{N})$; $|\bigcup_{\gamma<\alpha} T_\gamma| \leqslant \omega$ so there is a one-to-one map $\bigcup_{\gamma<\alpha} T_\gamma \xrightarrow{\varphi_\alpha} \omega$; define $f_\alpha : T \to \omega$ by

$$f_\alpha(\gamma) = \begin{cases} \varphi_\alpha(\gamma) & \text{when } \lambda(\gamma) < \alpha, \quad (3^\alpha) \\ \psi_\alpha(\gamma) & \text{when } \lambda(\gamma) = \alpha, \text{ i.e. } \gamma \in T_\alpha, \quad (2^\alpha) \\ \psi_\alpha(p_\alpha(\gamma)) & \text{when } \lambda(\gamma) > \alpha. \quad (1^\alpha) \end{cases}$$

Define $A_\alpha^n \equiv f_\alpha^{-1}\{0, \ldots, n-1\}$. This defines μ.

- Fix α. $\bigcup_n A_\alpha^n = f_\alpha^{-1}(\omega) = T$ $(= \omega_1)$.
- Take $f \in \omega^{\omega_1}$. $\exists n_0$ $(|f^{-1}(n_0)| > \omega)$. $\forall \alpha \in f^{-1}(n_0)$, let $W_\alpha \equiv f_\alpha^{-1}\{0, \ldots, n-1\} \cap T_\alpha$. $\mathcal{W} = \{W_\alpha\}_{\omega_1}$ satisfies (c), therefore $\exists \alpha \neq \beta$ $(x \in W_\alpha,\ y \in W_\beta \Rightarrow x,\ y$ incomparable), so $A_\alpha^{n_0} \cap A_\beta^{n_0}$ is finite: $\gamma \in A_\alpha^{n_0} \cap A_\beta^{n_0}$ means γ satisfies $(1^\alpha \vee 2^\alpha \vee 3^\alpha) \wedge (1^\beta \vee 2^\beta \vee 3^\beta) = [\ldots] \vee (3^\alpha \wedge 3^\beta)$. For each of the 2×3 cases in $[\ldots]$, there are only $< \omega$ such γ, immediately. There is no γ satisfying $3^\alpha \wedge 3^\beta$: if γ does, $\lambda(\gamma) > \alpha, \beta$, and $\psi_\alpha(p_\alpha(\gamma)), \psi_\beta(p_\beta(\gamma)) \in \{0, \ldots, n_0 - 1\}$ and thus (the condition of (c)) $p_\alpha(\gamma)$ and $p_\beta(\gamma)$ are incomparable. But they are not, because $\downarrow\gamma$ is well ordered. $\square$

We turn to the extension of Theorem 4.1 suggested in the second sentence of Theorem 1.4. “Sum” (of spaces) means “disjoint union” (which is the categorical sum). We denote generically by Σ a sum of uncountably many non-void compact spaces.

Theorem 4.7.

(a) *If W is any uncountable discrete space, the $(C(\beta W), \tau^{\mathcal{C}})$ is not a topological group.*
(b) *If Y contains a Σ as a clopen set, then $(C(\beta Y), \tau^{\mathcal{C}})$ is not a topological group.*
(c) *If Y is paracompact, locally compact, and zero-dimensional, and if $(C(\beta Y), \tau^{\mathcal{C}})$ is a topological group, then Y is Lindelöf.*

Proof. We sketch the proof, omitting numerous details.

(i) Suppose G_1, G_2 are groups, with respective topologies t_i, and suppose $(G_1, t_1) \xrightarrow{\varphi} (G_2, t_2)$ is a continuous homomorphism. If φ is relatively open (i.e., open onto its range), and if (G_1, t_1) is a topological group, then so is $(\varphi(G_1), t_2|\varphi(G_1))$.

Any continuous map $A \xrightarrow{\mu} B$ creates a group homomorphism $C(B) \xrightarrow{\widetilde{\mu}} C(A)$ by composition: $\widetilde{\mu}(f) = f \circ \mu$.

Let $(A, \mathfrak{F}), (B, \mathfrak{G}) \in$ **LSpFi**. A continuous map $A \xrightarrow{\mu} B$ for which $[G \in \mathfrak{G} \Rightarrow \mu^{-1}(G) \in \mathfrak{F}]$ is a morphism of the category **LSpFi**, and we write $(A, \mathfrak{F}) \xrightarrow{\mu} (B, \mathfrak{G}) \in$ **LSpFi**. (See the Appendix.)

(ii) If $(A, \mathfrak{F}) \xrightarrow{\mu} (B, \mathfrak{G}) \in$ **LSpFi**, then $(C(B), \tau^{\mathfrak{G}}) \xrightarrow{\widetilde{\mu}} (C(A), \tau^{\mathfrak{F}})$ is a continuous homomorphism.

(iii) In (ii), $\widetilde{\mu}$ is relatively open if both

(a) $\forall T \in \mathfrak{F}_\delta\ \exists S_T \in \mathfrak{G}_\delta\ (\mu^{-1}(S_T) \subseteq T)$, and
(b) $\forall L \in \mathcal{K}(B)\ \forall \epsilon \in (0,1)\ \forall f \in C(\mu(A))$ with $|f| \leqslant \epsilon$ on $L \cap \mu(A)$, $\exists g \in C(B)$ with $[g|\mu(A) = f$ and $|g| \leqslant \epsilon$ on $L]$.

(iv) Let $V \xrightarrow{\sigma} W$ be continuous, with Čech–Stone extension $\beta V \xrightarrow{\mu} \beta W$. Then $(\beta V, \mathcal{C}_V) \xrightarrow{\mu} (\beta W, \mathcal{C}_W) \in$ **LSpFi**, so $(C(\beta W), \tau^{\mathcal{C}_W}) \xrightarrow{\widetilde{\mu}} (C(\beta V), \tau^{\mathcal{C}_V})$ is a continuous homomorphism.

(v) Let $V \xrightarrow{\sigma} W$, μ and $\widetilde{\mu}$, be as in (iv). If either (a) σ is an embedding with $\sigma(V)$ clopen in W, or (b) σ is onto, W is discrete, $\forall x \in W$ ($\sigma^{-1}(x)$ is compact), then both of (iii) (a) and (b) hold, so that $(C(\beta W), \tau^{\mathcal{C}_W}) \xrightarrow{\widetilde{\mu}} (C(\beta V), \tau^{\mathcal{C}_V})$ is a relatively open continuous homomorphism, so that

$$\big[\big(C(\beta W), \tau^{\mathcal{C}_W}\big) \text{ not a topological group} \Rightarrow \big(C(\beta V), \tau^{\mathcal{C}_V}\big) \text{ not a topological group}\big].$$

We prove Theorem 4.7(b) in three steps, using Theorem 4.1 in the first. The first step is Theorem 4.7(a).

(1) Suppose W is uncountable discrete. So there is $D = V \xrightarrow{\sigma} W$ as (v)(a). By (v) and Theorem 4.1, $(C(\beta W), \tau^{\mathcal{C}_W})$ is not a topological group.

(2) Consider a Σ, then the obvious $\Sigma = V \xrightarrow{\sigma} W$, W uncountable discrete, as (v)(b). By (v) and (1), $(C(\beta\Sigma), \tau^{\mathcal{C}})$ is not a topological group.

(3) (= Theorem 4.7(b)). Suppose Y contains a Σ as a clopen set. Let $\Sigma = V \xrightarrow{\sigma} W = Y$ be the embedding, as (v)(a). By (v) and (2), $(C(\beta Y), \tau^{\mathcal{C}})$ is not a topological group.

We prove Theorem 4.7(c) from (b).

If Y is paracompact not Lindelöf, then Y contains a “uniformly discrete” copy of D (see [15]). If Y is also locally compact and zero-dimensional, then the copy of D can be enlarged to a Σ which is clopen. Then the embedding $\Sigma \xrightarrow{\sigma} Y$ is as (v)(a), so by (v) and Theorem 4.7(b), $(C(\beta Y), \tau^{\mathcal{C}})$ is not a topological group. □

It seems likely that some hypotheses in Theorem 4.7, and (v) above, can be relaxed. For example, if (v)(a) is replaced by (a′) [σ is an embedding with $\sigma(V)$ a C^*-embedded zero-set], then (iii)(a) holds; but we do not know about (iii)(b).

5. Appendix. On the context for this paper

We explain some lines of thought which have generated our studies of $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$. These involve epimorphisms in a category **W** of l-groups, and the closely related monomorphisms in the topological category **LSpFi**.

LSpFi is the category of “Spaces with Lindelöf filters”: Objects are the $(X, \mathfrak{F})$ considered in this paper, and a morphism $(X, \mathfrak{F}) \xrightarrow{f} (Y, \mathfrak{G})$ is a continuous function with $(\forall G \in \mathfrak{G}, f^{-1}(G) \in \mathfrak{F})$. A monomorphism (monic) of **LSpFi** is a left-cancelable morphism, i.e., m with ($mf = mg \Rightarrow f = g$ ($f, g \in$ **LSpFi**)). These monics are given several descriptions in [3] and [12].

A one-to-one morphism is monic, but not conversely. To the present point:

Theorem 5.1. *(See [12].) Consider a surjection $(X, \mathfrak{F}) \overset{\mu}{\twoheadrightarrow} (Y, \mathfrak{G})$ in* **LSpFi**. *Define $\widetilde{\mu} : C(Y) \to C(X)$ as $\widetilde{\mu}(f) = f \circ \mu$; This is an algebraic embedding of $C(Y)$ in $C(X)$. These are equivalent: μ is monic in* **LSpFi**; *$\widetilde{\mu}(C(Y))$ is dense in $(C(X), \tau^{\mathfrak{F}})$; $\widetilde{\mu}(C(Y))$ is dense in $(C(X), \sigma^{\mathfrak{F}})$.*

Remarks 5.2. (a) In Theorem 5.1, the densities in $\tau^{\mathfrak{F}}$ and $\sigma^{\mathfrak{F}}$ are equivalent in spite of $[\tau^{\mathfrak{F}} \lneq \sigma^{\mathfrak{F}}$ for infinite $X]$.

(b) The systems $\widetilde{\mu}(C(Y))$ in Theorem 5.1 are sub-vector-lattices of $C(X)$ containing 1, which are uniformly complete. Any such subset $A \subseteq C(X)$ is of the form $\widetilde{\mu}(C(Y))$: Form the topological quotient $X \overset{\mu}{\twoheadrightarrow} Y$ defined by $[\mu(x_1) = \mu(x_2)$ iff $a(x_1) = a(x_2)\ \forall a \in A]$, and give Y the quotient filter; then $\widetilde{\mu}(C(Y)) = A$.

(c) There is more to the subject of "monics in spaces with filter" [6,3]. **SpFi** (*sans* **L**) has objects $(X, \mathfrak{F})$, where $\mathfrak{F}$ consists of dense and merely open sets. There is a contravariant adjunction **SpFi** $\rightleftarrows$ **Frm**, the latter the category of completely regular frames; and μ is monic in **SpFi** iff its image is epic in **Frm**, but description of these morphisms [17] are not completely satisfactory. Under the adjunction, **LSpFi** and Lindelöf frames correspond, and the situation comes into sharper focus, e.g., Theorem 5.1 and [12,3,5], and see the following discussion about l-groups.

W is the category of archimedean lattice-ordered groups with distinguished weak order-unit, with unit-preserving l-group homomorphisms. Each $G \in |\mathbf{W}|$ has its Yosida representation $G \subseteq D(YG)$ as a point-separating l-group of continuous $[-\infty, +\infty]$-valued functions on certain compact Hausdorff YG, with all $g^{-1}((-\infty, +\infty))$ dense ($g \in G$). Let $\mathfrak{F}_G = \{g^{-1}((-\infty, +\infty)) \mid g \in G\}$. So $SYG \equiv (YG, \mathfrak{F}_G) \in$ **LSpFi**, and we have functor $SY : \mathbf{W} \to$ **LSpFi** (not onto), with the feature: φ is epic in **W** (i.e., right-cancelable) iff $SY\varphi$ is monic in **LSpFi**. (See [3].)

For the express purpose of studying epics in **W**, [2] puts a topology on each $G \in |\mathbf{W}|$, denoted there τ^G and called the "epi-topology", by exactly the procedure of Section 2 here, viewing $G \subseteq D(YG)$: First consider $S \in (\mathfrak{F}_G)_\delta$ and define compact-open τ_S on G *via* basic τ_S-neighborhoods $U(g, K, \epsilon) = \{f \in G \mid |f - g| \leqslant \epsilon$ on $K\}$ (compact $K \subseteq S$); then $\tau^G = \bigwedge\{\tau_S \mid S \in (\mathfrak{F}_G)_\delta\}$ on G.

Theorem 5.3. *(See 2.6 and 5.3 of [2].) Let H be a divisible sub-**W**-object of G. The inclusion $H \leqslant G$ is **W**-epic iff H is τ^G-dense in G.*

Similarly, we can define the compact-zero topologies σ_S ($S \in (\mathfrak{F}_G)_\delta$) on G, then $\sigma^G = \bigwedge\{\sigma_S \mid S \in (\mathfrak{F}_G)_\delta\}$, and show the additional equivalence in Theorem 5.3 "H is σ^G-dense in G", which fact we will spew out from our explanation of how Theorem 5.1 is a legitimate generalization of Theorem 5.3, which follows.

Let G^* be the sub-**W**-object of G of those $g \in G \subseteq D(YG)$ which are bounded functions. By the Stone–Weierstrass Theorem, G^* is uniformly dense in $C(YG)$. The relation between the topologies τ^G and σ^G on G, and the topologies $\tau^{\mathfrak{F}_G}$ and $\sigma^{\mathfrak{F}_G}$ on $C(YG)$, defined from the filter, is evident:

$$\tau^G|G^* = \tau^{\mathfrak{F}_G}|G^*; \qquad \sigma^G|G^* = \sigma^{\mathfrak{F}_G}|G^*,$$

where, e.g., $\tau^G|G^*$ denotes the subspace topology on G^*.

Let $H \leqslant G$. Then, $H^* \leqslant G^*$, and, it is easy to see that:

$$H \text{ is } \tau^G\text{-dense in } G \quad \textit{iff} \quad H^* \text{ is } \tau^G|G^*\text{-dense in } G^*;$$

$$H \text{ is } \sigma^G\text{-dense in } G \quad \textit{iff} \quad H^* \text{ is } \sigma^G|G^*\text{-dense in } G^*.$$

Now label the inclusion e: $H \leqslant G$. Applying the functor SY, we obtain the **LSpFi**-surjection $\mu \equiv SYe : (YG, \mathfrak{F}_G) \twoheadrightarrow (YH, \mathfrak{F}_H)$. Then, as in Theorem 5.1, we have the injection $\widetilde{\mu} : C(YH) \to C(YG)$. Note again that G^* (respectively, H^*) is uniformly dense in $C(YG)$ (respectively, $C(YH)$).

Then, a little thought about uniform limits, and τ^G and σ^G neighborhoods, reveals:

$$H^* \text{ is } \tau^G|G^*\text{-dense in } G^* \quad \textit{iff} \quad \widetilde{\mu}\big(C(YH)\big) \text{ is } \tau^{\mathfrak{F}_G}\text{-dense in } C(YG),$$

and likewise for σ^G and $\sigma^{\mathfrak{F}_G}$.

Combining these thoughts with Theorems 5.3 and 5.1, we have

Corollary 5.4. *Let H be a divisible sub-**W**-object of G, with μ, $\widetilde{\mu}$ etc. as above. The following are equivalent*

(1) *$H \leqslant G$ is epic in **W**.*
(2) *H is τ^G-dense in G.*
(3) *H is σ^G-dense in G.*
(4) *$\mu : (YG, \mathfrak{F}_G) \twoheadrightarrow (YH, \mathfrak{F}_H)$ is monic in **LSpFi**.*
(5) *$\widetilde{\mu}(C(YH))$ is $\tau^{\mathfrak{F}_G}$-dense in $C(YG)$.*
(6) *$\widetilde{\mu}(C(YH))$ is $\sigma^{\mathfrak{F}_G}$-dense in $C(YG)$.*

[2, 5.6, §6, 8.8] raises the question: For $G \in |\mathbf{W}|$, when is the epi-topology τ^G a topological group topology on G? This is the motivation for the present paper, and the sequel "Topological group criterion for $C(X)$ in compact-open-like topologies, II". We now complete the explanation of how the present development addresses this.

Proposition 5.5. *Let $G \in |\mathbf{W}|$. The following are equivalent.*

(1) *On G, τ^G (respectively, σ^G) is a group topology.*
(2) *On G^*, $\tau^G|G^*$ (respectively, $\sigma^G|G^*$) is a group topology.*
(3) *On $C(YG)$, $\tau^{\mathfrak{F}_G}$ (respectively, $\sigma^{\mathfrak{F}_G}$) is a group topology.*

Corollary 5.6.

(a) *On $C(\lambda D(\omega_1)) = C$, τ^C and σ^C are group topologies.*
(b) *On $C(D(\omega_1)) = C$, τ^C and σ^C are not group topologies.*

Corollary 5.6 follows from Proposition 5.5, and Sections 3, 4 (as does the generalization of (b), using Theorem 4.7).

Proof of Proposition 5.5. It is evident that, if A is a subgroup of B, and t is a group topology on B, then $t|A$ is a group topology on A. Thus (1) $\Rightarrow$ (2), and (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) with a little thought about uniform limits and the form of neighborhoods of 0.

(2) $\Rightarrow$ (1): (i) Basic neighborhoods of 0 (in the four topologies involved) are convex ($|f| \leqslant |g|$ and $g \in U \Rightarrow f \in U$).

(ii) If basic neighborhoods U, V of 0 satisfy $V^+ + V^+ \subseteq U$, then $V + V \subseteq U$ (since $f, g \in V \Rightarrow |f|, |g| \in V^+$, and $|f + g| \leqslant |f| + |g| \in V^+ + V^+ \subseteq U$, so $f + g \in U$).

(iii) For any $E \subseteq YG$, $g \in G^+$, $\epsilon \in (0, 1)$, $g|E = 0$ (respectively, $g|E \leqslant \epsilon$) iff $g \wedge 1|E = 0$ (respectively, $g \wedge 1|E \leqslant \epsilon$). Thus

(iv) For a basic neighborhood W of 0 in G (for either τ^G or σ^G) and $g \in G^+$, $g \in W$ iff $g \wedge 1 \in W$.

Now let U be a basic neighborhood of 0 in G (for τ^G or σ^G). We want another, V, with $V^+ + V^+ \subseteq U$. Assume (2): $\exists V$ $(V^* + V^* \subseteq U^*)$. If $|f|, |g| \in V^+$, then $f \wedge 1, g1 \in V^*$. Now $(f + g) \wedge 1 \leqslant f \wedge 1 + g \wedge 1 \in V^* + V^* \subseteq U^*$, so $(f + g) \wedge 1 \in U$ since U is convex, thus $f + g \in U$. □

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